Circle Squaring with Jordan Measurable Pieces

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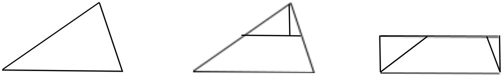
Equidecomposability

- $A, B \subseteq \mathbb{R}^k$ are equidecomposable ($A \sim B$):
  - $\exists\ A = A_1 \sqcup \cdots \sqcup A_n$
  - $\exists\ B = B_1 \sqcup \cdots \sqcup B_n$

  st $\forall\, i\ \ \exists$ isometry γ_i with $B_i = \gamma_i(A_i)$
- Example: 
- Wallace 1807, Bolyai 1832, Gerwien 1832: equi-area polygons are dissection congruent
  - Tarski'24: equidecomposable
  - Exercise: $[0,1] \sim [0,1)$

Banach-Tarski Paradox

- $\mathbb{B}^k$:= unit ball in $\mathbb{R}^k$
- Banach-Tarski Paradox'24: $\mathbb{B}^3 \sim \mathbb{B}^3 \sqcup \mathbb{B}^3$
- Hausdorff 1914: paradoxes on sphere $\mathbb{S}^{k-1}$, $k \geq 3$
- Proof idea:
  - $\mathrm{SO}(3) \supseteq$ free group $\mathbb{F}_2 = \langle a, b \rangle$
  - $S_a :=$ {irreducible words starting with a}, etc
  - $a^{-1}.S_a = S_a \cup S_b \cup S_{b^{-1}} \cup \{\emptyset\}$
- Axiom of Choice

Some consequences

- $\Rightarrow$ Banach-Tarski'24: $A, B \subseteq \mathbb{R}^k$, $k \geq 3$, bounded, non-empty interior $\Rightarrow$ $A \sim B$
- $\Rightarrow$ No non-trivial finitely-additive isometry-invariant mean on all bounded subsets of $\mathbb{R}^k$, $k \geq 3$
- Tomkowicz-Wagon'16: *"The Banach-Tarski Paradox"*
- "Constructive" equidecompositions ?

Some σ-algebras

- Borel σ-algebra $\mathcal{B} := \sigma(\text{open sets})$
- $\mathcal{L} = \{\text{Lebesgue measurable}\} := \sigma(\mathcal{B} \cup \{\text{null sets}\})$
- $\mathcal{T} = \{\text{Baire measurable}\} := \sigma(\mathcal{B} \cup \{\text{meager sets}\})$

Baire measurable equidecompositions

- Dougherty-Foreman'92: $A, B \in \mathcal{T}(\mathbb{R}^k)$, $k \geq 3$, bounded, non-empty interior $\Rightarrow$ $A \sim_{\mathcal{T}} B$
  - Marks-Unger'16: simpler proof
- $\Rightarrow$ No non-trivial finitely-additive isometry-invariant mean on $\{X \in \mathcal{T}(\mathbb{R}^k) : \text{bounded}\}$, $k \geq 3$

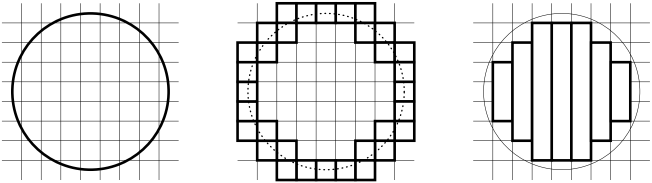
Lebesgue measurable pieces in $\mathbb{R}^k$, $k \geq 3$

- Necessary conditions for $A \sim_{\mathcal{L}} [0,1]^k$:
  - $A \in \mathcal{L}$
  - A is bounded
  - $\lambda(A) = 1$
  - finitely many copies of A cover $[0,1]^k$
- Grabowski-Máthé-P. $\geq$'21: sufficient for $k \geq 3$
- $\Rightarrow$ Margulis'82: $\forall$ isometry-invariant finitely-additive mean on $\mathcal{L} \cap \{\text{bounded}\}$ is $\mathrm{Constant} \cdot \lambda$
- Open: Can we replace $\mathcal{L}$ by $\mathcal{B}$ here ?

$\mathbb{R}^k$ with $k \leq 2$

- Banach'23: $A \sim B$, Lebesgue measurable $\Rightarrow$ $\lambda(A) = \lambda(B)$
  - Impossible to double a disk
- Tarski's Circle Squaring Problem'25: Is disk equidecomposable to a square ?
  - von Neumann'29: Yes, with affine transformations
  - Dubins-Hirsh-Karush'63: No, with topological disks
  - Gardner'85: No, with a discrete subgroup of $\mathrm{Iso}(\mathbb{R}^2)$
  - Laczkovich'90: YES, using translations only

Upper Minkowski (box) dimension in $\mathbb{R}^k$



- $N_\varepsilon(X)$ $:=$ # of ε-grid cubes intersecting X
- $\dim_\square X$ $:=$ $\limsup_{\varepsilon \to 0} \frac{\log(N_\varepsilon(X))}{\log(1/\varepsilon)}$
  - Supremum of d st $N_\varepsilon(X) \leq (1/\varepsilon)^{d+o(1)}$ as $\varepsilon \to 0$
- E.g. $\dim_\square \partial \mathbb{B}^2 = 1$
- $\dim_\square \partial X < k$ $\Rightarrow$ well approximated by boxes
  - $\exists\, K \subseteq \mathbb{B}^2 \subseteq U$ with $\lambda(U \setminus K) < \varepsilon$, using $O(1/\varepsilon)$ boxes
  - Box: $[a_1, b_1) \times \cdots \times [a_k, b_k)$
- $\dim_\square \partial X < k$ $\Rightarrow$ X is Jordan measurable
- X is Jordan measurable: bounded and $\lambda(\partial X) = 0$

More general result by Laczkovich

- Laczkovich'92: Bounded $A, B \subseteq \mathbb{R}^k$, $k \geq 1$
  - $\dim_\square \partial A, \dim_\square \partial B < k$
  - $\lambda(A) = \lambda(B) > 0$

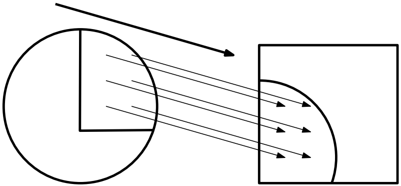
  $\Rightarrow$ $A \sim B$, using translations only
- Laczkovich'93,03: Cannot replace $\dim_\square$ by Hausdorff dimension, etc
- "Constructive" circle squaring ?

Constructive versions (using translations)

- Assume: bounded $A, B \subseteq \mathbb{R}^k$, $k \geq 1$, $\dim_\square \partial A, \dim_\square \partial B < k$, $\lambda(A) = \lambda(B) > 0$
- Grabowski-Máthé-P.'17: $A \sim_{\mathcal{L} \cap \mathcal{T}} B$
- Marks-Unger'17: $A, B \in \mathcal{B} \;\Rightarrow\; A \sim_{\mathcal{B}} B$
- Máthé-Noel-P. $\geq$'21:
  - $A \sim B$ with $\dim_\square \partial A_i < k$
  - $A, B \in \mathcal{B} \;\Rightarrow\; A \sim_{\mathcal{B}} B$ with $A_i \in \boldsymbol{B}(\boldsymbol{\Sigma}(\boldsymbol{B}(\text{boxes \& translates of } A \text{ and } B)))$
    - $\boldsymbol{\Sigma}$: countable unions
    - $\boldsymbol{B}$: Boolean combinations
- $\Rightarrow$ Circle squaring with $A_i \in \boldsymbol{B}(F_\sigma\text{-sets})$ and $\dim_\square \partial A_i < 1.985$
- Open: analogous Borel results for e.g. $SO(3) \curvearrowright \mathbb{S}^2$

Equidecompositions via graph matching

- Isometries $\gamma_1, \ldots, \gamma_n$
- Bipartite graph $\mathcal{H}$:
  - $V := A \sqcup B$
  - $E := \{xy \in A \times B : \exists\, i\ \gamma_i(x) = y\}$

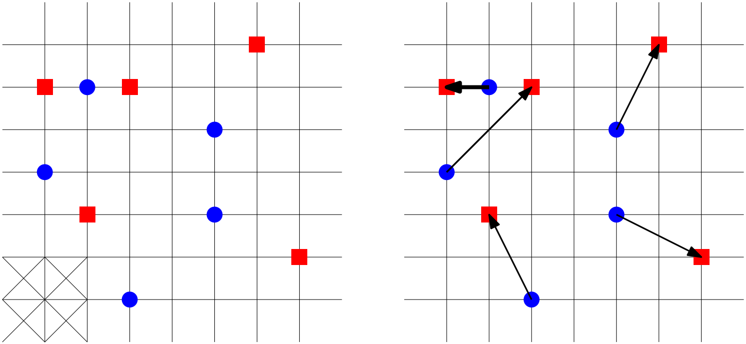


- Matching $\mathcal{M} \subseteq E \iff$ disjoint $A_1, \ldots, A_n \subseteq A$ st $\gamma_1(A_1), \ldots, \gamma_n(A_n) \subseteq B$ are disjoint
- $\exists$ perfect matching $\Rightarrow A \sim B$
- Each $A_i \in \mathcal{A} \Rightarrow A \sim_{\mathcal{A}} B$

Choosing translations

- Work on the torus $\mathbb{T}^k := \mathbb{R}^k/\mathbb{Z}^k$ (i.e. modulo 1)
- Fix $d = d(A, B)$
- Random $\mathbf{x}_1, \ldots, \mathbf{x}_d \in \mathbb{T}^k$
- Large $M = M(A, B, \mathbf{x}_1, \ldots, \mathbf{x}_d)$
- Vectors $\mathcal{V} := \{n_1\mathbf{x}_1 + \cdots + n_d\mathbf{x}_d : \mathbf{n} \in \{-M, \ldots, M\}^d\}$
- Recall $\mathcal{H} := (A \sqcup B, \{(\mathbf{u}, \mathbf{u} + \mathbf{v}) \in A \times B : \mathbf{v} \in \mathcal{V}\})$
- $\mathcal{G} := \big(\mathbb{T}^k, \{\,\{\mathbf{u}, \mathbf{u}+n_1\mathbf{x}_1+\cdots+n_d\mathbf{x}_d\} : \mathbf{n} \in \{-1, 0, 1\}^d\}\,\big)$
- Components of $\mathcal{G}$: $(3^d - 1)$-regular copies of $\mathbb{Z}^d$
- Aim: Bijection $\phi : A \to B$ with $\mathrm{dist}_{\mathcal{G}}(\mathbf{u}, \phi(\mathbf{u})) \leq M$

Local picture for $d = 2$ and $M = 2$



Discrepancy bounds by Laczkovich'92

- Discrete N-cube:

$$Q_{\mathbf{u},N} := \left\{ \mathbf{u} + \sum_{i=1}^{d} n_i \mathbf{x}_i : \mathbf{n} \in \{0, \ldots, N-1\}^d \right\}$$

- A is a box $\Rightarrow$ a.s. $\exists C$ $\forall$ discrete N-cube Q

$$\big|\, |A \cap Q| - \lambda(A) N^d \,\big| \le C \log^{k+d+1} N$$

- $\dim_{\square} A < k \Rightarrow$ a.s. $\forall$ discrete N-cube Q

$$\big|\, |A \cap Q| - \lambda(A) N^d \,\big| \le O(N^{d-1-\Omega(1)})$$

- $\Rightarrow$ $\mathcal{H}$ satisfies Hall's marriage condition
- Rado'49: $\exists$ perfect matching

Constructing a maximal matching $\mathcal{M}$

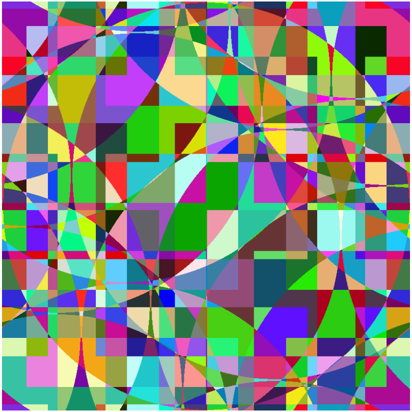
- Order $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \dots\}$
- $\mathcal{H} = (A \sqcup B, \{xy \in A \times B : \exists i \; x + \mathbf{v}_i = y\})$
- Greedy Algorithm:
  - $A_1 := A \cap (B - \mathbf{v}_1)$
  - $B_1 := A_1 + \mathbf{v}_1$
  - $A_2 := (A \setminus A_1) \cap ((B \setminus B_1) - \mathbf{v}_2)$
  - $B_2 := A_2 + \mathbf{v}_2$
  - $A_3 := (A \setminus (A_1 \cup A_2)) \cap ((B \setminus (B_1 \cup B_2)) - \mathbf{v}_3)$
  - ...
- $A_i, B_i \in \boldsymbol{B}$(translates of A and B)

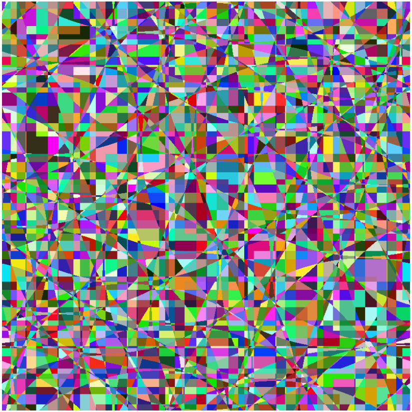
Maximal r-discrete sets

- $X \subseteq \mathbb{T}^k$ is r-discrete: $\forall\, x \neq y$ in X $\ \mathrm{dist}_{\mathcal{G}}(x, y) > r$
- Kechris-Solecki-Todorcevic'99: $\exists$ Borel maximal r-discrete set
- Box $[0, \varepsilon_r)^k \subseteq \mathbb{T}^k$ is r-discrete
- $(1/\varepsilon_r)^k$ translates cover $\mathbb{T}^k$
- $\exists$ maximal r-discrete $X \in \boldsymbol{B}$(boxes)

Local rules

- *r*-local rule: Boolean combination of A, B shifted by $\sum_{i=1}^{d} n_i \mathbf{x}_i$ with $\mathbf{n} \in \{-r, \ldots, r\}^d$
- Venn diagrams for $d = 2$, $r = 1$ and $r = 2$:



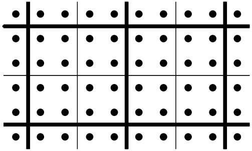


Lebesgue measurable circle squaring

- Discrepancy $\Rightarrow$ Stronger Hall's condition:

$$|N_{\mathcal{H}}(X)| \geq |X| + \Omega(|X|^{\frac{d-1}{d}}), \quad \forall \text{ finite } X \text{ in a part}$$

- Grabowski-Máthé-P.'17: $\forall$ matching $\mathcal{M}$ in $\mathcal{H}$ $\forall$ unmatched $\mathbf{x} \in A$ and $\mathbf{y} \in B$ $\exists$ an augmenting path from $\mathbf{x}$ to $\mathbf{y}$ of length $O(\mathrm{dist}_{\mathcal{H}}(\mathbf{x}, \mathbf{y}))$
- Augment matching in a dyadic-like way

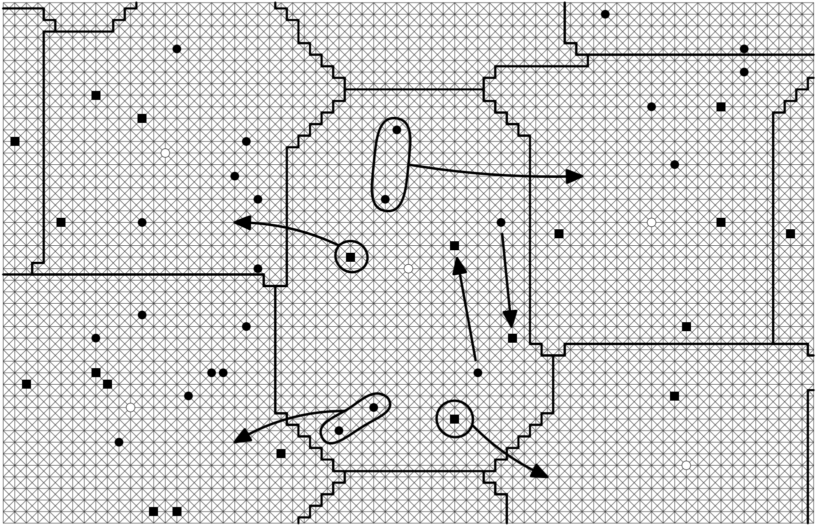


- $\mathcal{M}_i$: augment $\mathcal{M}_{i-1}$ inside level-i dyadic cubes
- Final $\mathcal{M} := \cup_{j=1}^{\infty} \cap_{i=j}^{\infty} \mathcal{M}_i$ matches A to B a.e.

Equidecompositions via flows

- Flow: $f : E(\mathcal{G}) \to \mathbb{R}$ st $f(x, y) = -f(y, x)$
- $f^{\mathrm{out}}(x) := \sum_y f(x, y)$
- AB-flow: $f^{\mathrm{out}} = -\mathbb{1}_A + \mathbb{1}_B$
- Marks-Unger'17: $\exists$ bounded integer-valued Borel AB-flow $\Rightarrow$ $A \sim_{\mathcal{B}} B$

Bijection from flow



Borel circle squaring (Marks-Unger'17)

- Bounded real-valued Borel AB-flow f_∞ in $\mathcal{G}$
  - Inductively construct f_i:
    - augment f_{i-1} in each 2^i-cube
    - average these over all 2^{id} choices of 2^i-grids
  - $f_\infty := \lim_{i\to\infty} f_i$
- Round f_∞ to integer-valued Borel flow g
- Marks-Unger'17: $\forall$ finite connected $S \subseteq \mathcal{G}$ one can make f_∞ integer on

$$\partial S := \{(\mathbf{x}, \mathbf{y}) \in E(\mathcal{G}) \cap (S \times S^c)\}$$

changing f_∞ only near ∂S, by at most 3^d

Jordan measurable pieces (Máthé-Noel-P.)

- $\{X \subseteq \mathbb{T}^k$: Jordan measurable$\}$ is not a σ-algebra
- High-level strategy:
  - Each locally-defined piece A_j grows
  - They exhaust A in measure
  - $\Rightarrow$ final A_j is Jordan measurable
  - Run Borel proof on $A \setminus \cup_j A_j$
- In terms of flows:
  - Construct partial integer-valued flows g_i, $i = 0, 1, \ldots$
  - g_i is local and is never overridden later

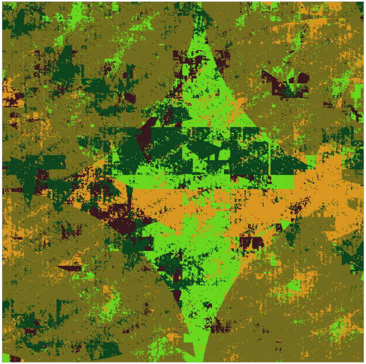
Jordan toast

- $r'_1 \ll r_1 \ll r'_2 \ll r_2 \ll \dots$
- Maximal r_i-discrete $X_i \in \boldsymbol{B}$(boxes)
- Near Voronoi cells $I_i := \{\mathbf{v} : \exists\, \mathbf{x} \in X_i \;\forall\, \mathbf{y} \in X_i \setminus \{\mathbf{x}\} \; \mathrm{dist}(\mathbf{v}, \mathbf{y}) \geq \mathrm{dist}(\mathbf{v}, \mathbf{x}) + r'_i\}$
- Leftover part has measure $\lambda(\mathbb{T}^k \setminus I_i) = O(r'_i / r_i)$
- "Swallow" components of $I_{<i}$ that come too close
- No two components of I_i are merged
- Lemma: when rounding ∂S, it is enough to know only approximation f_m instead of f_∞, $m = m(A, B, S)$

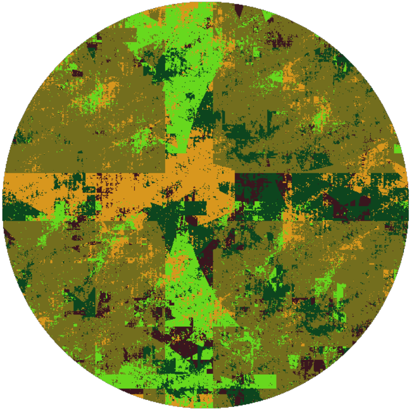
Open Problems

- Open: Bounded $A \sim B$, $\lambda(A) = \lambda(B) \Rightarrow A \sim B$ measurably ?
- Minimum number of pieces for circle squaring?
  - Laczkovich'92: *"a rough estimate"* is 10^{40}
  - Marks-Unger'17: $\leq 10^{200}$
  - Lower bound:
    - 3 (if rotations allowed)
    - 4 (for translations only)
- Banach-Tarski'24: 5 pieces for $\mathbb{B}^3 \sim \mathbb{B}^3 \sqcup \mathbb{B}^3$

Discrete circle squaring



580×580 torus



Thank you!