Subgroups of PL_oI which do not embed into F

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PL_oI: The group of all orientation preserving homeomorphisms of $[0,1]$ which are piecewise linear.

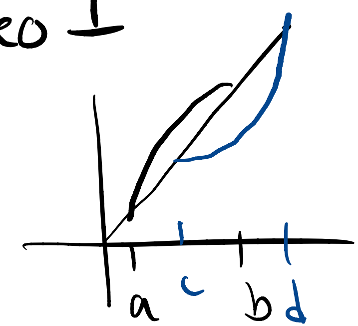
"Subgroups of PL_oI ... have been a source of groups with interesting properties in which calculations are practical." – Matt Brin

Thompson's Group F: All $f \in PL_oI$ which have breakpoints at dyadic rationals and whose slopes are powers of 2.

How to think about F:

Brin's Ubiquity Theorem If $G \leq PL_o I$, J is an orbital of J and some element of G reaches one end of J but not the other, then $F \hookrightarrow G$.

Example Suppose that $f, g \in \mathrm{Homeo}\, I$



$\mathrm{Supt}(f) = (a, b)$ $\quad$ $\mathrm{Supt}(g) = (c, d)$

$a < c < b < d$ and

$f(c) \geq g(b)$. Then $\langle f, g \rangle \cong F$.

General Program Understand the quasiorder of finitely generated subgroups of $PL_o I$ ordered by embeddability (homomorphic).

well quasiordered

Thesis This order should be highly structured (but complex!) below Thompson's group F; structure should completely break down above F.

Conjecture (Brin) Any finitely generated subgroup of $PL_o I$ either is embeddable into F or else contains F.

Conjecture (Brin, Sapir) If $G \leq PL_o I$ does not contain F, G is an elementary (amenable) group

Theorem (Bleak, Brin, M.) There are elementary, f.g. subgroups of F (G_ξ's are f.g. subgroups of F), groups G_ξ ($\xi < \varepsilon_0$) such that:

(1) G_0 is the trivial group and $G_{\xi+1} \cong G_\xi + \mathbb{Z}$

(2) For all $\xi, \eta < \varepsilon_0$, $G_\xi \hookrightarrow G_\eta$ iff $\xi \leq \eta$.

(3) Each G_ξ is elementary with class $\alpha_\xi < \varepsilon_0$ where $\sup_{\xi < \varepsilon_0} \alpha_\xi = \varepsilon_0$.

This talk Establish a criteria for when a subgroup of PL_oI does not embed into F.

Theorem (Lodha) The <u>Stein groups</u> $F_{p,q}$ do not embed into F if p,q are relatively prime.

Reason: The <u>groups of germs</u> have rank >1 and this is not possible in F.

Theorem (Hyde, M.) If $f,g \in PL_oI$ are an F-obstruction and $\varphi : \langle f,g \rangle \xrightarrow{1-1} PL_oI$ is a monomorphism, then $(\varphi(f), \varphi(g))$ is an F-obstruction.

Prop If $f, g \in F \leq PL_o I$, then f, g is not an F-obstruction.

Remark: $F_{p,q}$ contains an F-obstruction for each $p, q > 1$ with $\gcd(p,q) = 1$.

Cleary's "Golden ratio F" contains an F-obstruction.

Poincaré's Rotation Number

Suppose that γ is a homeomorphism of $\mathbb{R}/\mathbb{Z}$ and $\tilde{\gamma}$ is a lift. The rotation # of γ is the limit

$$\theta = \lim_{n \to \infty} \frac{\tilde{\gamma}^n(x) - x}{n}$$

modulo 1. Does not depend on x or $\tilde{\gamma}$.

Theorem (Poincaré) The rotation # being irrational implies that γ is topologically semiconjugate to a rotation by θ.

Theorem (Herman) If γ is PL, e.g. then γ is is in fact topologically conjugate to a rotation by its rotation #.

What is an F-obstruction?

Suppose $f, g \in PL_o I$ and $s \in I$ and $s < f(s) \leq g(s) < f(g(s)) = g(f(s))$

Define $\gamma : [s, g(s)) \to [s, g(s))$ by $\gamma(t) = g^m(f(t))$ where m is unique such that $g^m(f(t)) \in [s, g(s))$.

(m is unique and either 0 or -1).

This γ "is" a homeomorphism of the circle.

The rotation of γ is The <u>rotation</u>
<u># of f modulo g at s.</u>

f, g is an F-obstruction if for
some s, the rotation number of
f modulo g at s is irrational.

(Also symmetrize this so that if
f, g is an F-obstruction, so is
$f^{\pm 1}, g$, $f, g^{\pm 1}$, etc.)

$F \subseteq PL_o I$ doesn't contain F-obstructions

Theorem (Ghys-Sergiescu) Thompson's group T does not contain elements with irrational rotation #.

Analysis of 1-orbital F-obstructions

The first step is to show that if f, g is an F-obstruction and J is the orbital of (f, g) witnessing this, then there are dense $A, B \subseteq J$ such that if $a < b$, $a \in A$, $b \in B$, then $\exists h \in \langle f, g \rangle$ s.t. $\mathrm{Supt}(h) \cap J = (a, b)$.

A dichotomy theorem for subgroups of PL_oI

Suppose that $G \leq PL_oI$ and $J, K_0, \ldots, K_n$ are orbitals of G and G is <u>resolvable</u> on J. Then either:

(1) there is some $g \in G$ s.t. $\mathrm{supt}(g) \cap J \neq \phi$ and yet $\mathrm{supt}(g)$ is disjoint from K_i's.

(2) there is a G-equivariant monotone $\psi: K_i \twoheadrightarrow J$ for some $i = 0, \ldots, n$.

$$\psi(g(x)) = g(\psi(x))$$

Open Problems

1. If $G \leq PL_o I$ does not contain an F-obstruction and is finitely generated, must $G \hookrightarrow F$?

2. Suppose $G \leq PL_o I$ is finitely generated and does not embedd into F. Must G have an orbit which is somewhere dense.

3. Does $F^{+\mathbb{Q}} := \{ f^{t \mapsto t+q} : q \in \mathbb{Q} \} \leq \mathrm{Homeo}\, \mathbb{R}$ fail to embed into F?