A new proof of a theorem of Giordano, Putnam and Skau

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Caltech logic seminar

I. Topological dynamics on the Cantor space: some background.

Minimal homeomorphisms

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For any φ, there exists a closed $F \subseteq X$ such that $\varphi(F) = F$ and $\varphi_{\restriction F}$ is minimal; if F is infinite it is homeomorphic to X.

An example: the dyadic odometer

Define $\mathrm{Od}\colon \{0,1\}^\omega \to \{0,1\}^\omega$ as follows:

- If $x \neq 1^\infty$, set $n_x = \min\{i\colon x(i) = 0\}$ and

$$\mathrm{Od}(x)(i) = \begin{cases} 0 & \text{for all } i < n_x \\ 1 & \text{for } i = n_x \\ x(i) & \text{for all } i > n_x \end{cases}$$

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- If $x = 1^\infty$, set $\mathrm{Od}(x) = 0^\infty$

Then Od is a minimal homeomorphism; the associated equivalence relation is obtained from E_0 by gluing the classes of 0^∞ and 1^∞ together.

Kakutani–Rokhlin partitions

Fix a minimal φ, and a clopen $U \neq \emptyset$. For $x \in U$, set

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Then $n \colon U \to \mathbb{N}$ is continuous, hence has finite image F and each $U_i = \{x \in U \colon n(x) = i\}$ is clopen. Then one has

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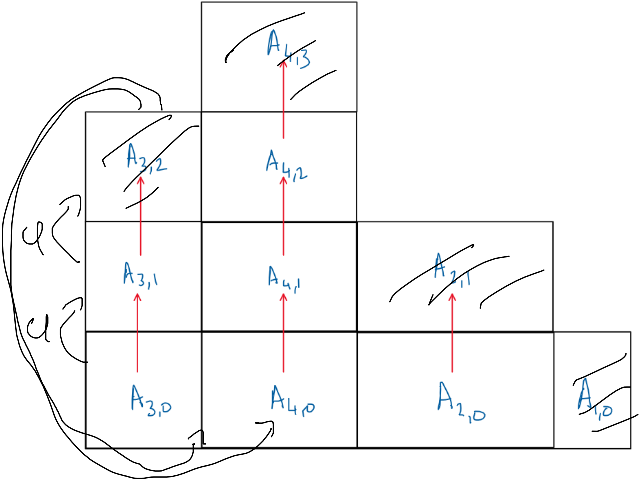
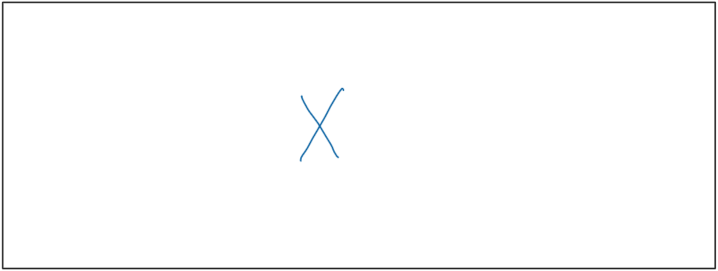
This is the archetype of a *Kakutani–Rokhlin partition*: a clopen partition $(A_{i,j})_{i \in F, j \in n_i}$ such that $\varphi(A_{i,j}) = A_{i,j+1}$ for all $j \in n_i - 1$.

An artist's rendition of a Kakutani–Rokhlin partition



Each atom not in the top is moved one level up by φ; the top is sent back to the base, and we cannot read any information about that on the partition.

Some Kakutani–Rokhlin partitions for the dyadic odometer



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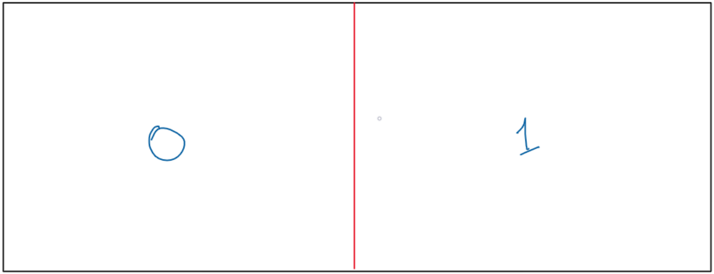

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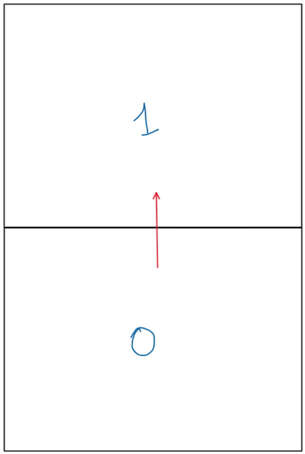

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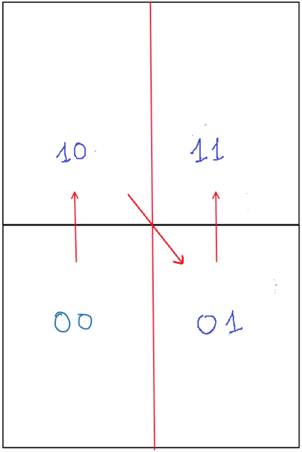

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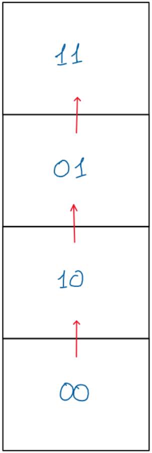

Sequences of Kakutani–Rokhlin partitions

When we shrink the basis we obtain a finer Kakutani–Rokhlin partition whose columns are obtained by "cutting and stacking" columns of the coarser one on top of each other.

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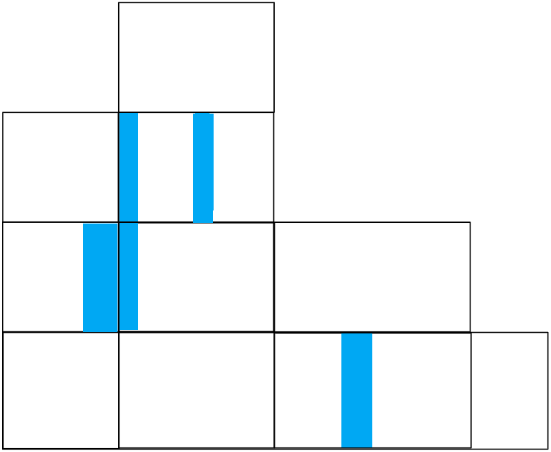
Such sequences are encoding a basis of neighborhoods of φ in $\mathrm{Homeo}(X)$.

Cutting a Kakutani-Rokhlin partition to make it compatible with a clopen set



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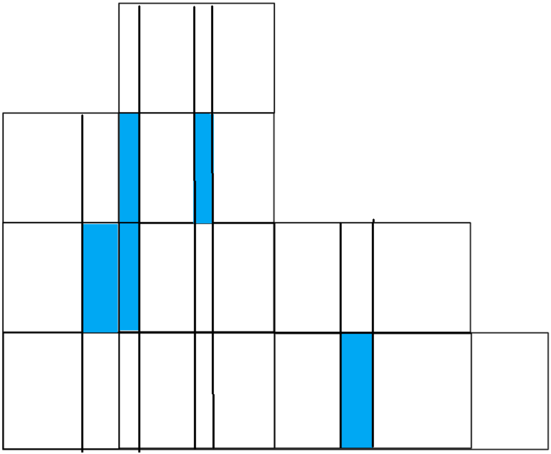

Locally finite groups attached to a $\mathbb{Z}$-action

Fix x, a minimal φ and a sequence of Kakutani-Rokhlin partitions $\mathcal{A}_n$ as above whose bases shrink to x.

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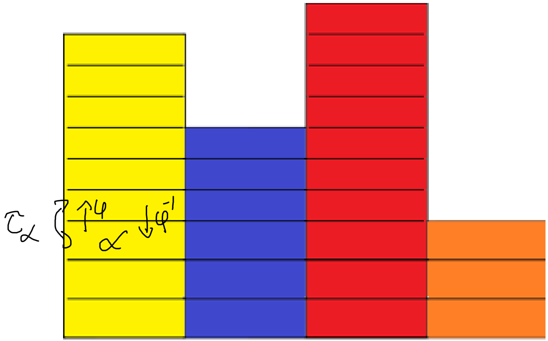
For the dyadic odometer (and bases shrinking to 0^∞) we obtain the group of dyadic permutations.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φ
α
φ^{-1}

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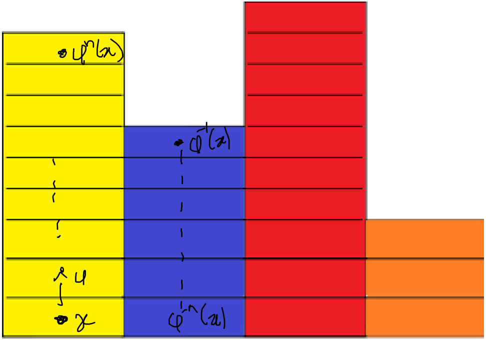

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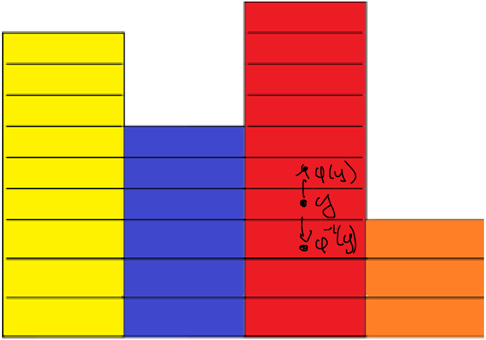

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- All other orbits for the actions of φ and $\Gamma_x(\varphi)$ on X are the same.

II. Orbit Equivalence.

Orbit equivalence

Definition

φ, ψ are *orbit equivalent* if there exists $g \in \mathrm{Homeo}(X)$ such that

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They also proved that φ and $\Gamma_x(\varphi)$ are OE for any minimal φ; this shows that E_0 and the relation induced from E_0 by gluing the classes of 0^∞ and 1^∞ are isomorphic.

Full groups I.

Definition

A subgroup $G \leq \mathrm{Homeo(X)}$ is a *full group* if : whenever $U_0, \dots, U_n$ is a clopen partition of X, $g_0, \dots, g_n$ are elements of G, and $g \in \mathrm{Homeo}(X)$ is such that $g_{|U_i} = g_{i|U_i}$ for all i, then $g \in G$.

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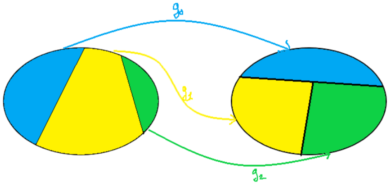

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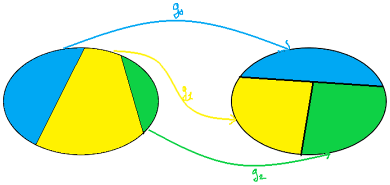

Definition

The *topological full group* of φ, denoted $[[\varphi]]$, is the smallest full group containing φ. It is a countable subgroup of $\mathrm{Homeo}(X)$.

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$[[\varphi]]$ consists of all elements of $[\varphi]$ such that $x \mapsto n_x$ is continuous. It contains each $\Gamma_x(\varphi)$.

One direction of the GPS theorem is easy

An orbit equivalence between φ and ψ is the same thing as a homeomorphism g such that $g[\varphi]g^{-1} = [\psi]$; and the set of all $[\varphi]$-invariant Borel probability measures coincides with $M(\varphi)$.

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Thus if φ and ψ are orbit equivalent via g then $g_*(M(\varphi)) = M(\psi)$.

The converse is more mysterious: two minimal homeomorphisms may preserve the same Borel probability measures yet have different orbits.

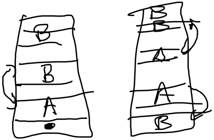
Theorem (Glasner–Weiss 1995)

Fix a minimal φ, and $x \in X$.

- For any two clopen A, B such that $\mu(A) < \mu(B)$ for all $\mu \in M(\varphi)$, there exists $g \in \Gamma_x(\varphi)$ such that $g(A) \subset B$.

$$\exists N \ \forall n \geq N \ \forall x \quad \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\varphi^i(x)}(A) < \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\varphi^i(x)}(B)$$

(compactness)



height $\geq N$

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The first item above is proved by a compactness argument, and the second follows from the first by back-and-forth.

Reformulating the Glasner–Weiss result in terms of full groups

For full groups G, H, one has $\overline{G} = \overline{H}$ iff for any clopen A, B

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Hence the second item on the previous slide implies that $\overline{[\varphi]} = \{g \in \mathrm{Homeo}(X) \colon \forall \mu \in \underbrace{M(\varphi)} \ g_* \mu = \mu\}$.

Supposed to encode $[\varphi]$,
now we know it captures $\overline{[\varphi]}$.

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But in general $\overline{[[\varphi]]} \neq \overline{[\varphi]}$. Key problem to prove GPS –

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We say that Γ is *saturated* if

$$\overline{\Gamma} = \{g \in \mathrm{Homeo}(X) \colon \forall \mu \in M(\Gamma)\ g_* \mu = \mu\}$$

Theorem (Krieger 1979)

Assume that Γ, Λ are two ample subgroups of $\mathrm{Homeo}(X)$ such that for any clopen U, V

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- It follows that $\Gamma_x(\varphi)$, $\Gamma_{x'}(\varphi)$ are conjugate for any two $x, x' \in X$.

($\overline{\Gamma_x(\varphi)} = \overline{[[\varphi]]} = \overline{\Gamma_{x'}[\varphi]}$)

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- It follows that $\Gamma_x(\varphi)$, $\Gamma_{x'}(\varphi)$ are conjugate for any two $x, x' \in X$.
- From Krieger's theorem, one easily obtains the particular case of the GPS theorem where φ, ψ are *saturated*, i.e. $\overline{[[\varphi]]} = \overline{[\varphi]}$, $\overline{[[\psi]]} = \overline{[\psi]}$; and similarly for orbit equivalence of saturated ample groups.

How to prove the GPS theorem?

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- A more recent "elementary" proof of Hamachi–Keane–Yuasa (2011) elaborates on ideas of Glasner–Weiss; it is quite long and technical.
- Based on the above discussion, we would like a proof, as elementary as possible, of the following fact: every minimal homeomorphism is orbit equivalent to a saturated minimal homeomorphism.

A refinement of Krieger's theorem

Given an ample group Γ acting minimally, say that K is Γ-*sparse* if K intersects each orbit in at most one point.

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Theorem

Let Γ, Λ be two ample groups acting minimally, and such that for any clopen U, V

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The proof is by back-and-forth (adapting Krieger's original argument).

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Then $\tilde{\Gamma}$ and Λ are OE by our refinement of Krieger's theorem, and Γ, $\tilde{\Gamma}$ are conjugate. So Γ is OE to a saturated ample group, and this proves that invariant measures are a complete invariant of OE for minimal ample groups.

GPS classification theorem for $\mathbb{Z}$-actions

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- Fix φ, and x, y belonging to different φ-orbits. Denote $\Gamma_{x,y}(\varphi) = \Gamma_x(\varphi) \cap \Gamma_y(\varphi)$. It is ample, acts minimally; $\Gamma_{x,y}(\varphi)$ and $\Gamma_x(\varphi)$ have the same invariant Borel probability measures. Hence they are OE.

orbits: $\mathcal{O}^+(x)$, $\mathcal{O}^-(x)$, $\mathcal{O}^+(y)$, $\mathcal{O}^-(y)$
all other Γ-orbits.

$\Gamma_x(\varphi)$ OE $\Gamma_{x,y}(\varphi)$

$\Gamma_{x,y}(\varphi)$ OE "$\Gamma_{x,y}(\varphi)$ + two orbits glued together"

$\Gamma_x(\varphi)$ OE $\Gamma_\varphi(x)$ + two (any) orbits glued together $\simeq R_\varphi$
(say $\mathcal{O}^+(x)$, $\mathcal{O}^-(x)$)

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- Fix φ, and x, y belonging to different φ-orbits. Denote $\Gamma_{x,y}(\varphi) = \Gamma_x(\varphi) \cap \Gamma_y(\varphi)$. It is ample, acts minimally; $\Gamma_{x,y}(\varphi)$ and $\Gamma_x(\varphi)$ have the same invariant Borel probability measures. Hence they are OE.
- So $\Gamma_x(\varphi)$ is OE to a relation obtained by gluing together two $\Gamma_x(\varphi)$-orbits. This is true for any two orbits by our refinement of Krieger's theorem: $\Gamma_x(\varphi)$ and φ are OE.

Thanks for your attention!

R_Γ + "gluing together some classes
according to some other
eq rel"

OE R_Γ