Local Entropy and Descriptive Complexity

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Overview

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Entropy

Entropy is a measures of disorder and chaos in a system.

Origin: Thermodynamics by Clausius

Statistical Mechanics: Boltzmann, Gibbs, Planck

Quantum Mechanics: von Neumann

Information Theory: Shannon

Dynamical Systems: Kolmogorov-Sinai (measure-theoretic)

Adler, Konheim and McAndrew (topological), Bowen and Dinaburg

Local Entropy

Local entropy theory is a culmination of deep results in dynamics, ergodic theory and combinatorics. Given a dynamical system with positive entropy, it gives, in some sense, the location of where the entropy resides. Moreover, it gives combinatorial tools to which can be used to study the location of entropy.

This seminal study was began by Blanchard in the 90's, after more than a decade of important contributions by many ended with a unified approach to the subject by Kerr-Li.

We refer reader to the survey paper by Glasner-Ye for the history of subject.

Applications

It is a powerful tool that can be applied in a variety of settings.

For example, D-Kato, using local entropy theory, showed positive entropy implies indecomoposability in the inverse limit of a dynamical systems (X, T) where X is G-like for some graph G.

Bernardes-D-Vermersch gave a simple proof of the fact that (X, T) has upe iff $(M(X), \tilde{T})$ has upe. Here $M(X)$ is the space of Borel probability measures on X and $\tilde{T}$ is the induced map on $M(X)$.

Dynamical Systems

A topological dynamical system (TDS) is simply (X, T), X compact metric and $T : X \to X$ continuous.

A measurable dynamical system (MDS) is simply $(X, \mathcal{B}, \mu, T)$, X a set, $\mathcal{B}$ a σ-algebra, μ a probability measure, and $T : X \to X$ measurable and measure preserving, i.e. $\mu(T^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}$.

Entropy (Measure-Theoretic)

Let $(X, \mathcal{B}, \mu, T)$ be a MDS.

Let $\mathcal{P}$ be a measurable partition of X. Then,

$$H_\mu(\mathcal{P}) = -\sum_{p \in \mathcal{P}} \mu(p) \ln(\mu(p))$$

If $\mathcal{P}$, Q are measurable partitions of X, then

$$\mathcal{P} \wedge \mathcal{Q} = \{p \cap q : p \in \mathcal{P}, q \in \mathcal{Q}.\}$$

Entropy (Measure-Theoretic) cont'd

The entropy of T with respect to partition $\mathcal{P}$ is given by

$$H_\mu(\mathcal{P}, T) = \liminf_n \frac{H_\mu(\mathcal{P} \wedge T^{-1}(\mathcal{P}) \wedge \ldots \wedge T^{-(n-1)}(\mathcal{P}))}{n}$$

And the entropy of T

$$H_\mu(T) = \sup_{\mathcal{P}} H_\mu(\mathcal{P}, T).$$

Entropy (Topological)

Let (X, T) be a TDS.

For an open $\mathcal{C}$ of X, $|\mathcal{C}|$ denotes least integer k such that there is a subcover of $\mathcal{C}$ with cardinality k which covers X.

The topological entropy of T with respect to partition $\mathcal{C}$ is given by

$$h(\mathcal{C}, T) = \liminf_n \frac{\ln |\mathcal{C} \wedge T^{-1}(\mathcal{C}) \ldots \wedge T^{-(n-1)}(\mathcal{C})|}{n}$$

And the entropy of T

$$h(T) = \sup_{\mathcal{C}} h(\mathcal{C}, T).$$

Full Shift Example

$\{0,1\}^\omega$; $\sigma(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$

$C = \{[0], [1]\}$

$\tau^{-1}(e) = \{ [0](1), [1](1)\}$

$\tau^{-2}(e) = \{ [0](2), [1](2)\}$

$C \vee T^{-1}(e) \ldots \vee T^{-(n-1)}(e) = \{ [\tau] : \tau \in \{0,1\}^n \}$

$H(\sigma) \geq \ln 2$

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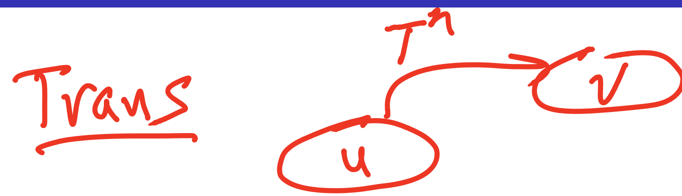
Some more defs

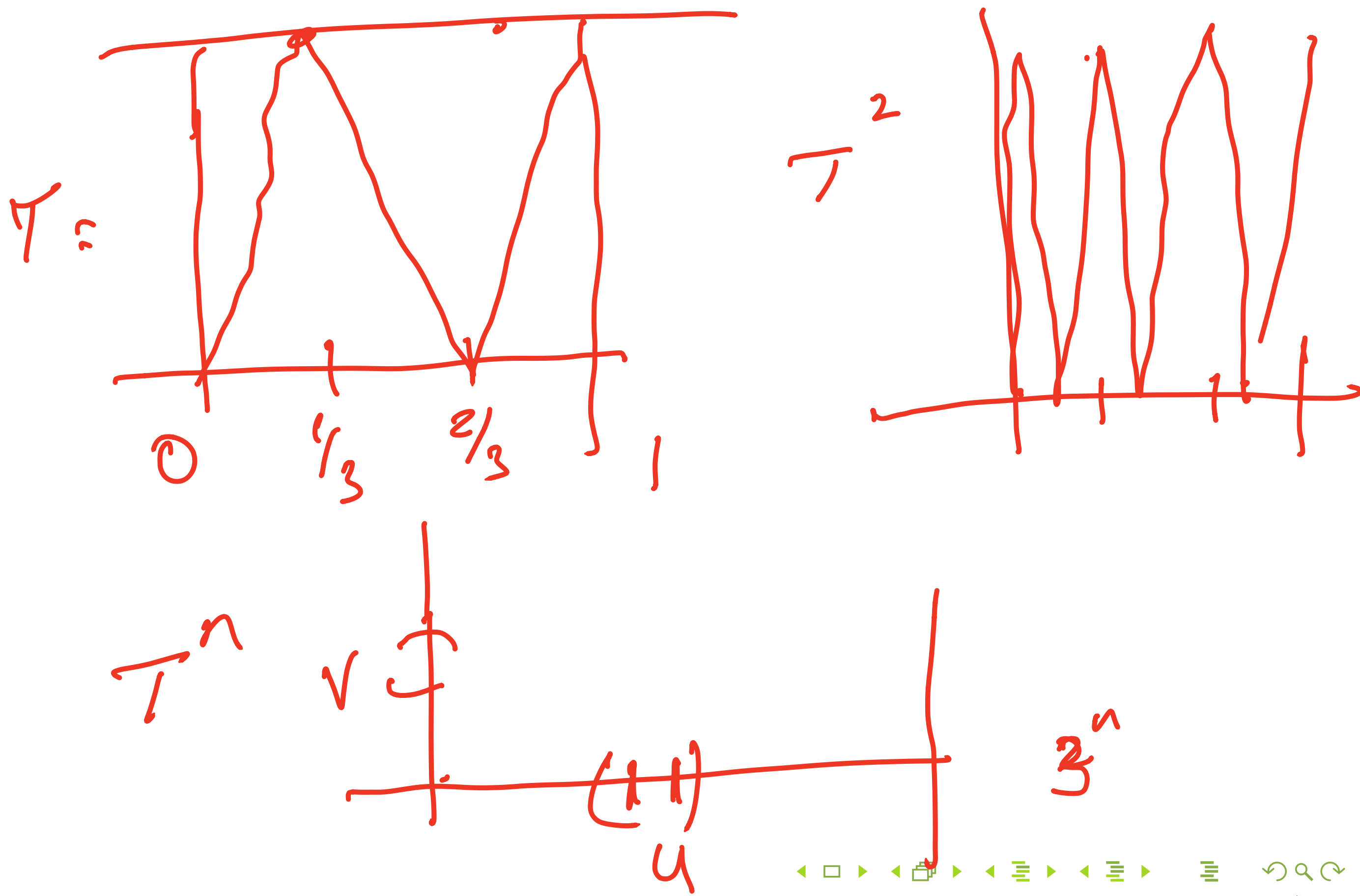


A TDS $T : X \to X$ is transitive if for all nonempty open U, V, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. This is equivalent to a dense orbit as long as X has no isolated points.

A TDS $T : X \to X$ is is weakly mixing if $T \times T$ is transitive.

A TDS $T : X \to X$ is mixing if for all nonempty open U, V, there exists $n \in \mathbb{N}$ such that $T^m(U) \cap V \neq \emptyset$ for all $m \geq n$.

A Mixing Example



Connection Between TDS and MDS

Theorem (Variational Principle)

Let (X, T) be a TDS and $M(T)$ the set of probability measures μ which are T-invariant.

$$h(T) = \sup_{\mu \in M(T)} H_\mu(T).$$

Connections, cont'd

Theorem (K-system)

Let $(X, \mathcal{B}, \mu, T)$ be a MDS. Then, the following are equivalent.

- *For each finite partition $\mathcal{P}$ of X, we have that $H_\mu(\mathcal{P}, T) > 0$.*
- *For each 2-element partition $\mathcal{P}$ of X, we have that $H_\mu(\mathcal{P}, T) > 0$.* upe
- *T has cpe, complete positive entropy, i.e. each non-trivial factor of T has positive entropy (in measure theoretic sense).*

Any of the above imply that T is strongly mixing (in measure theoretic sense.)

UPE and CPE

Blanchard defined what he called UPE (Uniform Positive Entropy) and CPE in topological dynamics to capture the above relationship.

A TDS (X, T) has upe (uniform positive entropy) if for any essential cover $\mathcal{C} = \{U, V\}$ we have that $h(\mathcal{C}, T) > 0$.

A TDS (X, T) has cpe (complete positive entropy) if each non-trivial factor of T has positive topological entropy.

Theorem (Blanchard)

Let (X, T) be a TDS. Then,

- *upe implies cpe*
- *upe implies topological weak mixing.*
- *There are cpe systems which are not upe.*

Independence Set

A set $I \subseteq \mathbb{N}$ has positive density if $\liminf_n \frac{|I \cap [1,n]|}{n>0} > 0$.

Given a TDS (X, T) and $\{U, V\} \subset X$, we say $I \subset \mathbb{N}$ is an independence set for $\{U, V\}$ if for all finite $J \subseteq I$, and for all $(Y_j) \in \prod_{j \in J} \{U, V\}$, we have that

$$\cap_{j \in J} T^{-j}(Y_j) \neq \emptyset.$$

Independence Pic

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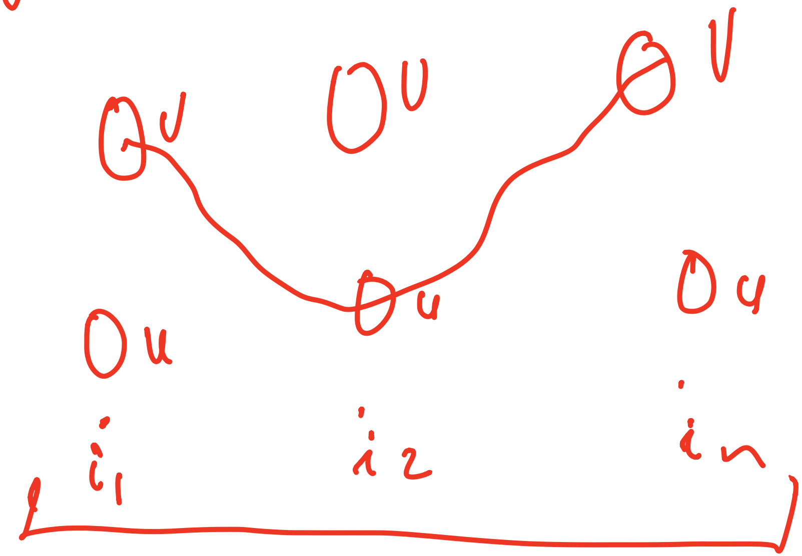
$T : X \to X$ $\{U, V\}$ open sets

$I \subseteq \mathbb{N}$ is an independent set

$\iff \forall F \subseteq I$ finite

$\bigcap_{j=1}^{n} T^{-i_j}(Y_j) \neq \emptyset$



$\in F$

IE-pair

Let (X, T) be a TDS. We say that $(x_1, x_2) \in X \times X$ is an **independence entropy pair (IE-pair) of (X, T)** if for every pair of open sets A_1, A_2, with $x_1 \in A_1$ and $x_2 \in A_2$, there exists an independence set for $\{A_1, A_2\}$ with positive density. The set of IE-pairs of (X, T) is denoted by $E(X, T)$.

Theorem (Kerr-Li)

Let (X, T) be a TDS.

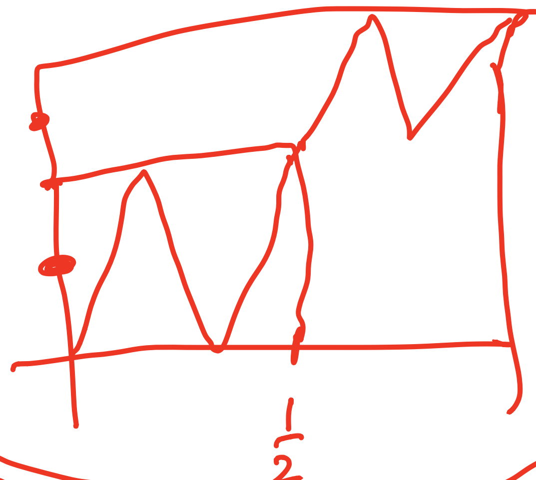
1. *(X, T) has positive entropy if and only if there exists $x \neq y \in X$ with $(x, y) \in E(X, T)$.*
2. *(X, T) has upe if and only if $E(X, T) = X \times X$.*

Theorem (Blanchard)

(X, T) has cpe if and only if the smallest closed equivalence relation containing $E(X, T) = X \times X$.

A CPE example which is not UPE

T is Mixing + u.p.e



T_2

$[0,\frac{1}{2}] \times [0,\frac{1}{2}]$

$[\frac{1}{2},1] \times [\frac{1}{2},1]$

$[0,1]$; $E = [0,\frac{1}{2}]^2 \cup [\frac{1}{2},1]^2 = E(T_2)$

Operator Γ (Barbieri, Garcia-Ramos)

Let X be a compact metric space and $E \subseteq X^2$.

We define E^+ as the smallest equivalence relation that contains E and $\Gamma(E) = \overline{E^+}$.

For a successor ordinal α,

$$\Gamma^\alpha(E) = \Gamma(\Gamma^{\alpha-1}(E))$$

and for limit ordinal α,

$$\Gamma^\alpha(E) = \overline{\cup_{\beta<\alpha}\Gamma^\beta(E)}$$

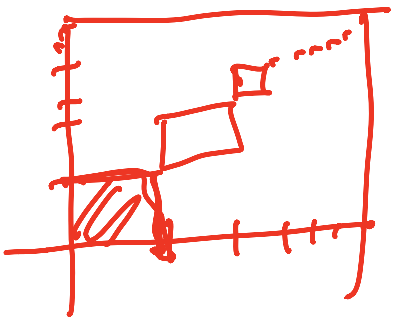
.

Γ-rank Examples

$I = [0,1]$

Ex 1

$$E = [0,\tfrac{1}{2}]^2 \cup [\tfrac{1}{2},\tfrac{3}{4}]^2 \cup [\tfrac{3}{4},\tfrac{7}{8}]^2 \cdots \cup \{(1,1)\}$$



0 1

$$E^+ = [0,1)^2 \cup \{(1,1)\}$$

$$\overline{E^+} = [0,1]^2$$

Ex 2 $K \subseteq [0,1]$ be ctble compact

$E = \bigcup J^2$, J contiguous to K. $\cup\, \Delta [0,1]$

Γ-rank(E) = Cantor B rank K

Alternate Formulation of CPE

Theorem (Barbieri, Garcia-Ramos)

A TDS (X, T) has cpe if and only if there exists a countable ordinal α such that $\Gamma^{\alpha}(E(X, T)) = X \times X$. The least such ordinal α is called the cpe rank of (X, T).

Recent Results

Theorem (Pavlov)

Th set of $\mathbb{Z}$ SFT with the cpe property is (lightface) Borel.

Theorem (Barbieri, Garcia-Ramos)

For every countable ordinal α, there exists a TDS (X, T) with cpe rank α where X is the Cantor space.

Theorem (Westrick)

cpe rank is a (lightface) Π_1^1-rank.

Proof is carried out using effective descriptive set theory.

Corollary

Let X be the Cantor space. The set of (X, T) cpe systems is Π_1^1 and not Borel.

Results, cont'd

Theorem (Westrick)

The set of $\mathbb{Z}^2$-SFT with the cpe property is (lightface) Π_1^1-complete.

Theorem (Salo)

The set of $\mathbb{Z}$ subshifts with the cpe property has arbitrarily high rank and hence is not Borel.

Borel Derivatives

We next recall the notion of derivatives and how it induces Π^1_1-ranks in a natural way. Let $\mathcal{K}(X)$ denote the space of all compact subsets of X endowed with the Hausdorff metric.

Definition

A map $D : \mathcal{K}(X) \to \mathcal{K}(X)$ is a **derivative** if the following holds:

$$D(A) \subseteq A \quad \& \quad A \subseteq B \implies D(A) \subseteq D(B).$$

Derivatives appear in a variety of contexts and they induce Π^1_1-ranks in a natural way. For a derivative D, let

$$D^0(A) = A$$
$$D^{\alpha+1} = D(D^\alpha(A))$$
$$D^\lambda(A) = \cap_{\beta<\lambda} D^\beta(A) \textit{ if } \lambda \textit{ is a limit ordinal.}$$

Derivatives, cont'd

Let $A \in \mathcal{K}(X)$. Then, there exists a countable ordinal α such that $D^\alpha = D^{\alpha+1}$. Moreover, we let $D^\infty(A) = D^{|A|_D}$, i.e., the stable part of A.

The following is an important theorem which relates derivatives to Π^1_1-ranks.

Theorem (Kechris-Louveau)

Let $D : \mathcal{K}(X) \to \mathcal{K}(X)$ be a Borel derivative and

$$C = \{A \in \mathcal{K}(X) : D^\infty(A) = \emptyset\}.$$

Then, C is Π^1_1 and $\varphi : C \to \omega_1$ defined by $\varphi(A) = |A|_D$ is a Π^1_1-rank on C.

Expansions

Definition

A map $E : \mathcal{K}(X) \to \mathcal{K}(X)$ is an **expansion** means that

$$A \subseteq E(A) \quad \& \quad A \subseteq B \implies E(A) \subseteq E(B)$$

For an expansion E, as earlier, we let

$$E^0(A) = A$$
$$E^{\alpha+1} = E(E^\alpha(A))$$
$$E^\lambda(A) = \overline{\cup_{\beta<\lambda} E^\beta(A)} \textit{ if } \lambda \textit{ is a limit ordinal.}$$

We let $|A|_{\mathrm{E}}$ be the least such α such that $\mathrm{E}^{\alpha+1} = \mathrm{E}^\alpha(A)$. Moreover, we let $\mathrm{E}^\infty(A) = \mathrm{E}^{|A|_{\mathrm{E}}}$, i.e., the stable part of A.

We have the following theorem whose proof is similar to the derivative case.

Theorem Known ??

Let X be a compact metric space and E be a Borel expansion on $\mathcal{K}(X)$ and let

$$C = \{A \in K(X) : E^{\alpha}(A) = X \text{ for some } \alpha\}.$$

Then, C is Π^1_1 and $\varphi : C \to \omega_1$ defined by $\varphi(A) = |A|_E$ is a Π^1_1-rank on C.

Borelness of $E(X, T)$, Γ

Proposition (D-Garcia-Ramos)

Let X be a compact metric space and $C(X, X)$ be the set of all continuous functions from X into X endowed with the uniform topology. Consider the mapping $E : C(X, X) \to K(X \times X)$ given by $E(T) = E(X, T)$. Then, E is a Borel map.

Proposition (D-Garcia-Ramos)

Let X be a compact metric space. Then, $\Gamma : K(X \times X) \to K(X \times X)$ defined by

$$\Gamma(A) = \overline{(A^+)}.$$

is a Borel map.

cpe rank is Π^1_1

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Introduction

Local Entropy

Derivatives, Expansions, Π^1_1-ranks

Main Results

Corollary

Let X be a compact metric space, C be the set all $T \in C(X,X)$ which satisfies CPE. Then, C is Π^1_1. Moreover, the map $\varphi : C \to \omega_1$ defined by $\varphi(T) = |E(X,T)|_\Gamma$ is a Π^1_1-rank on C.

When CPE(X) is Borel

$X = \{0,1\}^\omega$; CPE(X) is Π^1_1-complete

Theorem (D-Garcia-Ramos)

Among TDS with the shadowing property, the collection of cpe systems is Borel.

Theorem (D-Garcia-Ramos)

Among graph maps with topological mixing property, the collection of cpe systems is Borel.

Non-Borel Results

Theorem (D-Garcia-Ramos)

Let $d > 0$. The collection of TDSs on $[0,1]^d$ with cpe is Π^1_1-complete.

Theorem (D-Garcia-Ramos)

The collection of mixing TDS on the Cantor space with cpe is true Π^1_1.

Proof: CPE([0,1]) is Π^1_1-complete

Proposition (Hurewicz)

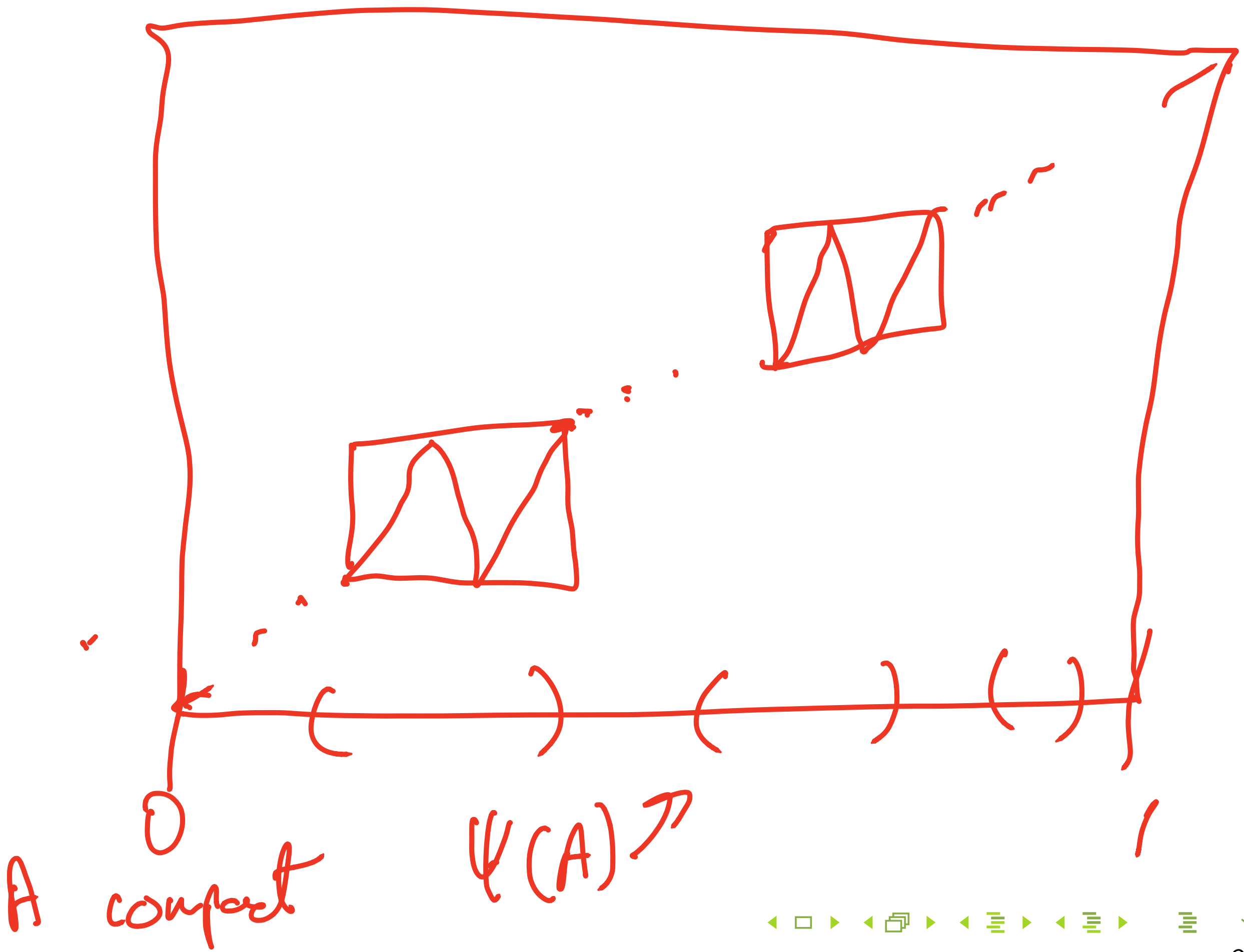
Let $I = [0,1]$ and $K(I)$ be the space of all nonempty closed subsets of I, endowed with the Hausdorff metric. Then,

$$\mathcal{Q} = \{A \in K(I) : A \text{ is countable}\}$$

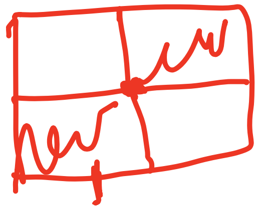
is Π^1_1-complete.

It suffices to construct a continuous function $\psi : K(I) \to C(I, I)$ such that $\psi(A)$ is CPE if and only if A is countable.

The assignment $\Psi(A)$, $A \in K(I)$.



Some Basic Facts



Fact

Suppose I is the union of two non-overlapping intervals I_1 and I_2 and $T : I \to I$ is such that $T(I_i) = I_i$.

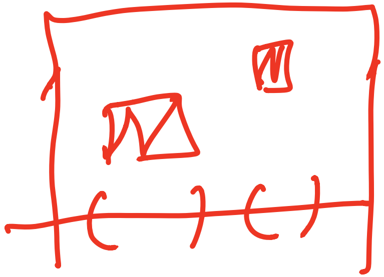
- *If x_i is in the interior of I_i, then (x_1, x_2) is NOT an IE-pair of T.*
- *If $I_i \times I_i \subseteq E(f)$, then $E(I, T)^+ = I$.*

More Basic Facts

Fact

Suppose $P \subseteq I$ is perfect, $T : I \to I$ is such that $T(J) = J$ for all intervals J contiguous to P. Then,

$$E(I,T) \subseteq \bigcup_{J \in \mathcal{C}(P)} (J \times J) \cup \Delta_P.$$



Proof on I

We now observe that $E(\psi(A))$, the set of IE-pairs of $\psi(A)$, is the union of $A \times A$ with $J \times J$ where J's range over the intervals contiguous to A.

Proof on I, cont'd

Furthermore, for countable ordinal α, $\Gamma^\alpha(E(\psi(A)))$ is simply the union of $A^\alpha \times A^\alpha$ with $J \times J$ where J's range over the set of intervals contiguous to A^α. From this it follows that $\Gamma^\infty(E(\psi(A)))$ is simply the union of $A^\infty \times A^\infty$ with $J \times J$ where J's range over the set of intervals contiguous to A^∞. Now from the basic facts about the Cantor-Bendixon derivativs, we have that $\psi(A)$ is CPE if and only if A is countable, completing the proof.

Some Explanations

Thank You!