Real Determinacy in Admissible Sets

Caltech Logic Seminar

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Outline

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- The game is played as follows: players I and II alternate turns playing elements of X. Player I plays on turns indexed by an even ordinal and Player II plays on turns indexed by an odd ordinal.
- After α-many turns, a sequence $x \in X^\alpha$ has been produced. Player I wins if $x \in A$; otherwise Player II wins.
- The game is *determined* if one of the players has a winning strategy.

Determinacy axioms

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- These have the form "all games of length α with moves in X and payoff in the pointclass Γ are determined."
  - Thus, there are three parameters in play: the length, the pool of possible moves, and the complexity of the games played.

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- Lower bound: one strong cardinal. Upper bound (strict): one Woodin cardinal.
- The purpose of this talk: study some of these axioms over some theories weaker than ZFC.

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  3. $\mathbb{R}$ exists.
- KP is strong enough to define L and perform various types of generalized recursion.

Determinacy axioms

- AD is the assertion that all games of length ω on $\mathbb{N}$ are determined.

Theorem (Woodin)

The following are equiconsistent:

1. $ZF + DC + AD$,
2. $ZFC +$ *there are infinitely many Woodin cardinals.*

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- Woodin's proof can easily be carried out in KP + DC + AD.
- Thus, KP + DC + AD implies the consistency of ZFC + {there are n Woodin cardinals: $n \in \mathbb{N}$}.
- Thus, KP and ZF have similar strength in the context of DC + AD (or just AD).

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- The proof is easy, but we will omit it.

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- One would expect that, as before, the strength of $\mathsf{KP} + \mathsf{AD}_{\mathbb{R}}$ is similar to that of $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$.

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Theorem

Suppose that there are ω^2 Woodin cardinals. Then, there is a transitive model of $KP + DC + AD_{\mathbb{R}}$ containing $\mathbb{R}$.

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- By a theorem of Neeman, if M_α exists, then all Π^1_1 games of length $\alpha \cdot \omega$ are determined.

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- For some additional perspective, suppose there are ω^2 Woodin cardinals and a measurable cardinal above them. Then, by a theorem of Steel, the model M_{ω^2} exists.
- This is the canonical model of ZFC with ω^2 Woodin cardinals.
- By a theorem of Neeman, if M_α exists, then all Π^1_1 games of length $\alpha \cdot \omega$ are determined.
- By relativizing, we see that if there are ω^2 Woodin cardinals below a measurable, then all analytic games of length ω^3 are determined.

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Theorem (Martin, Woodin)

Let $\alpha \geq \omega \cdot 2$ be a recursive wellordering which is provably wellfounded in ZFC. The following are equivalent over ZF + DC:

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- Over KP, however, this equivalence is not true. Determinacy axioms for longer games form a proper hierarchy.

Picture

Theories and their consistency strength. (1)

ZFC + "there are infinitely many Woodin cardinals"

|

Δ^1_1-Determinacy for games of length ω^2 on $\mathbb{N}$

|

ZC + "there are infinitely many Woodin cardinals"

|

$L_{\kappa^{\mathbb{R}}}(\mathbb{R}) \models AD$ $\equiv$ Σ^0_1-det for games of length ω^2 on $\mathbb{N}$

|

$L_{\omega_1}(\mathbb{R}) \models AD$ $\equiv$ σ-proj. det $\equiv$ Δ^0_1-det for games of length ω^2 on $\mathbb{N}$

| |

$L_1(\mathbb{R}) \models AD$ $\equiv$ PD (schema) $\sim$ ZFC + {there are n Woodins : $n \in \mathbb{N}$}

⋮

|

Δ^1_2-Determinacy $\equiv$ Σ^1_2-Determinacy $\equiv$ $\forall x\ M_1^{\#}$ exists

|

Σ^1_1-Determinacy $\equiv$ $\forall x\ x^{\#}$ exists

|

ZFC

|

ZC + Δ^1_1-Determinacy

|

ZC

Credits: Friedman, Martin, Harrington, Woodin, Steel, Müller, Schlicht, A.

Picture

Similar picture

(2)

ZFC + "there are ω^2 Woodin cardinals"

|

Δ^1_1-det of length ω^3

|

⋮

|

There is a transitive model $\mathbb{R} \in M \models KP + DC + AD_{\mathbb{R}}$ – Σ^0_1-det of length ω^3

|

$\sigma(\partial^{\mathbb{R}})$-det of length ω^2 – Δ^0_1-det of length ω^3

|

PD for games of length ω^2 – ZFC + {there are ωn Woodins : $n \in \mathbb{N}$}

|

⋮

|

Σ^1_1-det for games of length ω^2.

|

ZFC + "there are infinitely many Woodin cardinals"

Credits: Trang, Müller, A.

Picture

Picture from further curry.

$ZF + DC + AD_{\mathbb{R}}$ — $ZF + DC +$ "all countable games are determined"

$ZF + AD_{\mathbb{R}}$

$KP + AD_{\mathbb{R}} + \Sigma_1$ - Separation

↑

$ZF + AD + \{\Theta > \Theta_n : n \in \mathbb{N}\}$ - $ZFC + \lambda$ is a limit of Woodins $+ \{$there are n $<\lambda$-strong cardinals$: n \in \mathbb{N}\}$

↑

$ZF + AD + \Theta > \Theta_0$

↑

ZFC + "there are ω_1 Woodin cardinals"

⋮

↑

$KP + AD_{\mathbb{R}}^{\omega^{1+\alpha}}$, $KP + AD_{\mathbb{N}}^{\omega^{\alpha}}$] α provably recursive in ZFC

↑

$KP + AD_{\mathbb{R}}^{\omega^2}$, $KP + AD_{\mathbb{N}}^{\omega^3}$

↑

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↑

$ZF + AD$

↑

$KP + AD$

Can add DC

Credits: Neeman, Woodin, Steel, A.

- For the remainder of the talk, let us sketch the proof of the following theorem:

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- We first focus on the existence of a model of KP + $AD_{\mathbb{R}}$ from open determinacy of length ω^3. This requires reviewing the theory of Spector classes of relations and inductive definability.

Generalized quantifiers

- A *quantifier* on $\mathbb{R}$ is a non-empty collection of subsets of $\mathbb{R}$ closed under supersets but not equal to $\mathcal{P}(\mathbb{R})$. We write $Qx\, A(x)$ for $A \in Q$. pause
- Example:
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$$\Game^{\mathbb{R}} = \{A \subset \mathbb{R} : \text{Player I has a w.s. on the game of length } \omega \text{ on } \mathbb{R} \text{ with payoff } A\}$$

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Generalized quantifiers

- Note: Q is closed under supersets, so $Qx\, A(x)$ is equivalent to $\exists Y \in Q \forall x \in Y\, A(x)$. Using this triviality, we can make sense of expressions such as

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$$Qx_1\,Qx_2\,\ldots\,\phi(x_1, x_2, \ldots).$$

- Namely, this formula holds if and only if Player I has a winning strategy in the following game:
  - Player I begins by playing $Y_1 \in Q$,
  - Player II responds by playing $x_1 \in Y_1$,
  - Player I responds with $Y_2 \in Q$, etc.
  - After infinitely many rounds, Player I wins if and only if $\phi(x_1, x_2, \ldots)$ holds.

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We say that $\psi(x, X)$ is positive if X appears only positive in it.
- We are concerned with operators ϕ definable by positive second-order formulas $\psi(x, X)$ in the language of second-order arithmetic with constants for every real number and expanded by the quantifiers Q and $\check{Q}$.

Inductive definability

- Such positive operators can be iterated:

$$\phi^0 = \varnothing$$

$$\phi^\alpha = \phi\Big(\bigcup_{\beta<\alpha} \phi^\beta\Big)$$

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A set $A \subset \mathbb{R}$ is *Q-inductive* if $A = \{x : (x, a) \in \phi^\infty\}$ for some ϕ as above. We say that A is *Q-hyperprojective* if both A and $\mathbb{R} \setminus A$ are Q-inductive.

Spector classes

- We will need some more notions from generalized recursion theory.

Definition

A *Spector class* on $\mathbb{R}$ is an $\mathbb{R}$-parametrized collection of subsets of $\mathbb{R}$ closed under finite conjunctions and disjunctions, $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$, containing all projective sets, and having the prewellordering property.

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The Q-inductive sets are precisely those sets defined by a formula of the form

$$Qx_1\,\check{Q}x_2\,\exists x_3\,\forall x_4\,Qx_5\,\check{Q}x_6\,\ldots\,A(x_1, x_2, x_3, \ldots),$$

where A is projective.

Companion theorems

- Finally, we will need one of the companion theorems of Moschovakis:

Theorem (Moschovakis)

Let Γ be a Spector class on $\mathbb{R}$. Then, there is an admissible set M with $\mathbb{R} \in M$ and such that $\mathcal{P}(\mathbb{R}) \cap M = \Gamma \cap \check{\Gamma}$.

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- Consider the quantifier $\Game^{\mathbb{R}}_{\omega^2}$ consisting of all sets of reals A such that Player I has a winning strategy for the game on A with moves in $\mathbb{R}$ and length ω^2.
- Given $A \subset \mathbb{R}^2$, we write $\Game^{\mathbb{R}}_{\omega^2} A$ for $\{y \in \mathbb{R} : \{x : (x, y) \in A\} \in \Game^{\mathbb{R}}_{\omega^2}\}$.

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- Given $A \subset \mathbb{R}^2$, we write $\Game^{\mathbb{R}}_{\omega^2} A$ for $\{y \in \mathbb{R} : \{x : (x, y) \in A\} \in \Game^{\mathbb{R}}_{\omega^2}\}$.
- Write $\Game^{\mathbb{R}}_{\omega^2}\Sigma^0_1$ for the pointclass of all sets of the form $\Game^{\mathbb{R}}_{\omega^2} A$, with A open.

Proof sketch

- Lemma 1. The pointclass of all $\Game^{\mathbb{R}}$-inductive sets is contained in $\Game^{\mathbb{R}}_{\omega^2}\Sigma^0_1$.

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  - By Aczel's characterization, $\Game^{\mathbb{R}}$-inductive sets can be defined by a formula of the form $\Game^{\mathbb{R}} x_1 \breve{\Game}^{\mathbb{R}} x_2 \ldots \psi(x_1, x_2, \ldots)$, where ψ is projective.

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  - This formula has a specific semantics, but naively, we should be allowed to replace each game quantifier by an infinite string of real quantifiers. Thus, we obtain a definition of a given $\Game^{\mathbb{R}}$-inductive set by a formula of the form

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- Indeed, the converse of the lemma is true. We shall not prove that, but it will be used as well in the future.

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- Lemma 2. Suppose that all open games on of length ω^3 on $\mathbb{N}$ are determined. Then, all $\Game^{\mathbb{R}}_{\omega^2}\Sigma^0_1$-games of length ω^2 on $\mathbb{N}$ are determined.

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  - Thus, we can consider a game in which players play ω^2 many turns and then they are required to play the game given by the formula

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Combining the two lemmata: if all open games of length ω^3 are determined, then all games of length ω^2 on $\mathbb{N}$ with $\Game^{\mathbb{R}}$-inductive payoff are also determined.

Proof sketch

- Lemma 3. Suppose that $\Game^{\mathbb{R}}$-hyperprojective games of length ω^2 on $\mathbb{N}$ are determined. Then, every $\Game^{\mathbb{R}}$-hyperprojective game of length ω on $\mathbb{R}$ has a $\Game^{\mathbb{R}}$-hyperprojective winning strategy.

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  - Proof idea: First, one adapts Moschovakis' argument for showing that inductive sets have inductive scales in order to show that, under the hypotheses of the lemma, $\Game^{\mathbb{R}}$-hyperprojective sets have $\Game^{\mathbb{R}}$-hyperprojective scales. This requires (the proof of) Martin's theorem on the propagation of scales under the real-game quantifier.

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  - Then, one adapts the proof of Moschovakis' Third Periodicity Theorem to prove the lemma. This requires the scale property, as well as the fact that $\Game^{\mathbb{R}}$-hyperprojective relations can be uniformized by $\Game^{\mathbb{R}}$-hyperprojective functions (this follows from the existence of scales).

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- Suppose that open games of length ω^3 are determined. By the first two lemmata, all games of length ω^2 on $\mathbb{N}$ with $\Game^{\mathbb{R}}$-inductive payoff are also determined.
- By the third lemma, every $\Game^{\mathbb{R}}$-hyperprojective game of length ω on $\mathbb{R}$ has a $\Game^{\mathbb{R}}$-hyperprojective winning strategy.
- Let M be the companion model of the $\Game^{\mathbb{R}}$-hyperprojective sets obtained from Moschovakis' theorem. Then, the sets of reals in M are precisely the $\Game^{\mathbb{R}}$-hyperprojective sets. Thus, for each game in M of length ω on $\mathbb{R}$, there is a strategy in M. Therefore, $M \models \mathsf{AD}$.

Proof sketch

- Let us finish by sketching the argument for the converse. Let M be a transitive model of $\mathsf{KP} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$ such that $\mathbb{R} \in M$. We claim that all open games of length ω^3 are determined.

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- First, we need a stronger determinacy hypothesis in M; namely that all games of length ω^2 with moves in $\mathbb{N}$ are determined in M. This is proved using the uniformization property for sets in M.
- Thus, every $\Game^{\mathbb{R}}$-hyperprojective game of length ω^2 on $\mathbb{N}$ is determined.

Proof sketch

- We now need the following determinacy transfer theorem:

Theorem

Let α be a countable limit ordinal with $\omega^2 \leq \alpha$. Let Γ be an ω-parametrized pointclass containing all recursive sets and satisfying the prewellordering property. Suppose that Γ is closed under recursive substitution, finite unions and intersections, and the quantifier $\breve{\Game}^{\mathbb{N}}_{\alpha}$ for games of length α on $\mathbb{N}$. Suppose moreover that all games of length α with moves in $\mathbb{N}$ and payoff in $\Gamma \cap \breve{\Gamma}$ are determined.
Then, all games of length α with moves in $\mathbb{N}$ and payoff in Γ are determined.

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Then, all games of length α with moves in $\mathbb{N}$ and payoff in Γ are determined.

- The theorem is an extension of a determinacy transfer theorem due to Kechris and Solovay, and its proof is a very simple modification of Kechris and Solovay's proof.

Proof sketch

- Its consequence of relevance to us is that from the determinacy of all $\Game^{\mathbb{R}}$-hyperprojective games of length ω^2 on $\mathbb{N}$, we can conclude the determinacy of all $\Game^{\mathbb{R}}$-inductive games of length ω^2 on $\mathbb{N}$.

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- Hence, we can conclude the determinacy of all games of length ω^2 with moves in $\mathbb{N}$ and payoff in $\Game^{\mathbb{R}}_{\omega^2}\Sigma^0_1$.

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- Hence, we can conclude the determinacy of all games of length ω^2 with moves in $\mathbb{N}$ and payoff in $\Game^{\mathbb{R}}_{\omega^2}\Sigma^0_1$.
- To finish, we need to show that this implies open determinacy for games of length ω^3.

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- The idea is as follows: Let G be an open game of length ω^3 for which Player I does not have a winning strategy. Divide G into infinitely many blocks $G_1, G_2, \ldots$, of length ω^2 and consider each of them a separate game.

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- Observe that Player I does not have a winning strategy for H_1, because this would induce a winning strategy for G.

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- Suppose that the auxiliary game is determined in favor of Player II. Then, by playing G according to the strategy of H_1, after ω^2 turns, a real x is produced from which Player I does not have a winning strategy for G.

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- The point is that this sequence is a winning play for Player II in G. This is because the game is open, so if Player I were to win, she would do so at some bounded stage, but we argued that this was impossible.

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- We have just described a winning strategy for Player II in G.

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