Borel combinatorics fail in HYP

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Caltech Logic Seminar

Joint with Henry Towsner & Rose Weisshaar Supported by DMS-1854107

October 27, 2021



Prisoner hat game

- \bullet Infinitely many prisoners in a line, order type ω
- Each wears a red or blue hat
- Each sees the hats ahead of their own
- No one sees their own hat or previous hats
- Starting at the back, each tries to guess their own hat color
- They win if they make at most one mistake
- They are allowed to agree on a strategy beforehand
- Can they win?



Prop. (Folklore) There is no Borel winning strategy for the prisoners. *Proof.*

- Suppose \mathcal{R} is a Borel winning strategy.
- Then \mathcal{R} is measurable (has property of Baire).
- *R* measurable (has property of Baire) ⇒
 there is *τ* ∈ {*R*, *B*}^{<ω} such that prisoner 0 says *R* for 99% (co-meager many) of *S* ≻ *τ*.

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 there is *τ* ∈ {*R*, *B*}^{<ω} such that prisoner 0 says *R* for 99% (co-meager many) of *S* ≻ *τ*.
- Then for some S', prisoner 0 says R on both $\tau RS'$ and $\tau BS'$.
- Let $S_R = B\tau RS'$ and $S_B = B\tau BS'$
- Prisoner 0 is wrong if the complete sequence of hats is S_R or S_B .
- So prisoners 1 through $|\tau|$ must guess correctly on S_R and S_B .
- Prisoner $|\tau| + 1$ hears $B\tau$ and sees S' in both cases.
- So prisoner $|\tau| + 1$ is wrong on one of S_R or S_B .

Questions

- Are these different proofs?
- What theorems of Borel combinatorics can be proved by measure, category, both, or neither?

Some approaches

- Seek measure-theoretic and Baire-categoric results (DST)
- What set existence axioms do the proofs use? (Reverse mathematics)

Prop (Folklore). Using AC, the prisoners have a winning strategy in the prisoner hat game.

Proof. The prisoner's winning strategy is

- Fix beforehand a well-ordering of the reals.
- Each prisoner looks at the sequence of hats ahead of her and lets x be the least real that eventually agrees with the hat sequence.
- This strategy ensures that every prisoner picks the same real.
- The first prisoner states the parity of the errors x makes on the sequences of hats.
- Every subsequent prisoner can now deduce her own hat color (using all previous prisoner answers).

Second order arithmetic

- Most math can be carried out in second order arithmetic (SOA).
- In SOA, there are two kinds of objects, natural numbers and subsets of natural numbers (infinite bit sequences)
- Everything else is coded. For example, a Borel set *B* is given by a (code for a) well-founded, countably branching ∩/∪/clopen-labeled tree describing how to make *B*.



• The axioms of SOA, including the axioms of Peano Arithmetic for the natural numbers and various set existence axioms, suffice for most mathematics outside set theory.

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"When the theorem is proved from the right axioms, the axioms can be proved from the theorem." (Friedman 1968)

- Suppose Axiom A is used to prove Theorem T in SOA.
- Fix a base theory, some small fragment of SOA strong enough so T makes sense, but weaker than A.
- If *T* and the base theory together imply *A*, then *A* is necessary for proving *T*.
- The usual base theory is RCA₀, which roughly captures constructive mathematics.

What if T does NOT imply A?

- Showing *T* does not imply *A* requires a *separation* a model of SOA in which *T* holds but *A* does not.
- The SOA axioms do not guarantee the first-order part to be the true natural numbers.
- An ω -model of SOA is a model in which the first-order part is the true natural numbers.
- To specify an ω -model of SOA, we just give the second-order part, a subset $\mathcal{M} \subseteq 2^{\omega}$.
- ω -models are simplest and often used for separations.
- Important example: $\mathcal{M} = HYP = \Delta_1^1 \subseteq 2^{\omega}$.

The strongest non-constructive theorems/axioms being used in our proofs:

- Every Borel set is measurable
- Every Borel set has the property of Baire
- Arithmetic Transfinite Recursion

We want to compare the strength of these axioms, and ask if they are needed for theorems such as

- No Borel winning strategy in the prisoner hat game
- A Borel 2-regular acyclic graph with no Borel 2-coloring
- A Borel bipartite 3-regular graph with no Borel perfect matching
- etc.

What base theory should be used?

Borel set membership

Suppose we have a Borel set B and want to know if $X \in B$. (B is coded by a $\cap/\cup/$ clopen-labeled tree $S \subseteq \omega^{<\omega}$)

There is an inductive "procedure":

 $X \in B \iff \begin{cases} X \in B & \text{if } B \text{ is a basic open set or its complement} \\ \exists n[X \in B_n] & \text{if } B = \bigcup_n B_n \\ \forall n[X \in B_n] & \text{if } B = \bigcap_n B_n. \end{cases}$

One step is arithmetic, and the recursion has transfinite depth.

The axiom of Arithmetic Transfinite Recursion (ATR₀) roughly states that a procedure such as the above has a well-defined output, namely an *evaluation map* $f : S \to \{0, 1\}$ which indicates X's membership status in all subtrees of S.

Over RCA₀,

- (DSFW '21) ATR₀ is equivalent to the statement that for every well-founded ∩/∪/clopen-labelled tree *S*, there is an *X* which has an evaluation map in *S*.
- ATR₀ proves that every Borel set is measurable
- ATR₀ proves every Borel set has the property of Baire.

If ATR_0 is the base, our axioms cannot be distinguished.

Borel sets without ATR₀

Definition.¹ A Borel set coded by *S* is *completely determined* if for every $X \in 2^{\omega}$, there is an evaluation map for *X* in *S*.

Definition. A formula ϕ of $L_{\omega_1,\omega}$ is *completely determined* if there is a function f: Subformulas $(\phi) \rightarrow \{T, F\}$ which evaluates the formula. $L_{\omega_1,\omega}$ -CA₀ states: for every sequence $\langle \phi_n \rangle$ of c.d. formulas of $L_{\omega_1,\omega}$, the sequence $\langle f_n \rangle$ of evaluation maps exists.

Prop. Over $L_{\omega_1,\omega}$ -CA₀: complements, countable unions, countable intersections, and continuous pre-images of c.d. Borel sets are c.d. Borel.

Definition

- Let CD-PB be the principle "Every completely determined Borel set has the property of Baire."
- Let CD-M be the principle "Every completely determined Borel set is measurable."

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¹Astor, Dzhafarov, Montalban, Solomon, & W, 2020

Theorem. (ADMSW '20, W '21) Both CD-PB and CD-M are strictly weaker than ATR_0 .

Fact. Neither CD-PB nor CD-M implies the other. Thus, our two proofs of "no CD-Borel prisoner hat strategy" use different set-existence axioms.

Principles slightly weaker than ATR₀



Theories of hyperarithmetic analysis

Just below ATR_0 we find the theories of hyperarithmetic analysis.

A set of principles is a theory of hyperarithmetic analysis (THA) if

- all of its ω -models are hyperarithmetically closed $(X \in \mathcal{M} \implies \Delta_1^1(X) \subseteq \mathcal{M})$ and
- HYP(Y) is an ω -model for each $Y \in 2^{\omega}$. $(HYP(Y) = \Delta_1^1(Y) \subseteq 2^{\omega})$.

Theorem. (ADMSW '20) Every ω -model of CD-PB contains a Δ_1^1 -generic. Thus CD-PB is not a THA because CD-PB fails in *HYP*.

Theorem. (W '21, ADMSW '20) Every ω -model of CD-M contains a Δ_1^1 -random. Thus CD-M is not a THA because CD-M fails in *HYP*.

Question. Does "no CD-Borel winning strategy for the prisoner hat game" follow from $L_{\omega_1,\omega}$ -CA₀? Does it hold in *HYP*?

Completely determined Borel sets in HYP

From now on, fix the ω -model $\mathcal{M} = HYP$.

What are the completely determined Borel sets in HYP?

- Truly Borel sets (given by codes S that are actually well-founded)
 - Always c.d., with $f_X \leq_T X^{(\alpha)}$ for some fixed $\alpha \approx \rho(S)$
 - Always measurable
 - Always have Baire property
- Pseudo-Borel sets (given by codes *S* that are actually ill-founded, but have no *HYP* path)
 - We only consider the c.d. ones
 - In HYP, may not be measurable or have Baire property
 - No bound on the jumps needed to compute an evaluation map.

If *B* is c.d. pseudo-Borel given by code *S*, *B*-membership is Δ_1^1 (in SOA):

$$X \in B \iff \exists f[X \in_f B] \qquad X \notin B \iff \exists f[X \notin_f B]$$

α -recursion theory

Let α be any admissible ordinal (e.g. ω_1^{ck} , the least uncomputable ordinal)

Consider the initial segment L_{α} of Gödel's constructible universe L.

A subset $A \subseteq L_{\alpha}$ is called α -c.e. if A is $\Sigma_1(L_{\alpha})$. That is, there is a Σ_1 formula ϕ in the language of set theory such that

$$x \in A \iff L_{\alpha} \models \phi(x)$$

An α -c.e. set can be understood as the result of a meta-computation of length α because

$$L_{\alpha} \models \phi(x) \iff (\exists \beta < \alpha) L_{\beta} \models \phi(x).$$

A subset $A \subseteq L_{\alpha}$ is called α -computable if A is $\Delta_1(L_{\alpha})$.

Recall that $L_{\omega_1^{ck}} \cap 2^{\omega} = HYP$.

The statements $\exists f[X \in_f B]$ and $\exists f[X \notin_f B]$ are each $\Sigma_1(L_{\omega_i^{ck}})$.

So $X \in B$ is $\Delta_1(L_{\omega_1^{ck}})$.

Recall that $L_{\omega_1^{ck}} \cap 2^{\omega} = HYP$.

The statements $\exists f[X \in_f B]$ and $\exists f[X \notin_f B]$ are each $\Sigma_1(L_{\omega_1^{ck}})$.

So $X \in B$ is $\Delta_1(L_{\omega_1^{ck}})$.

Theorem. (Towsner, Weisshaar, W.) For any $A \subseteq HYP$, TFAE.

There is a completely determined Borel code for A in HYP.
A is ω₁^{ck}-computable.

First example

Theorem (TWW). In *HYP*, there is a completely determined Borel well-ordering of the reals.

Proof. For any $x \in HYP$, let α_x be the least ordinal such that

$$x \leq_{\mathcal{T}} \emptyset^{(\alpha_x)}$$

and let e_x be the least number such that

$$x = \Phi_{e_x}^{\emptyset^{(\alpha_x)}}$$

The ordering we desire is

$$x < y \iff lpha_x < lpha_y$$
 OR $(lpha_x = lpha_y$ and $e_x < e_y)$

This ordering is clearly ω_1^{ck} -computable. (We can tell whether x < y uniformly in $(x \oplus y)^{(\alpha_x + \alpha_y + 2)}$) **Theorem** (TWW). In *HYP*, the prisoners have a completely determined Borel winning strategy in the prisoner hat game.

Proof. The prisoner's winning strategy is

- Fix beforehand a well-ordering of the reals.
- Each prisoner looks at the sequence of hats ahead of her and lets x be the least real that eventually agrees with the hat sequence.
- This strategy ensures that every prisoner picks the same real.
- The first prisoner states the parity of the errors x makes on the sequences of hats.
- Every subsequent prisoner can now deduce her own hat color (using all previous prisoner answers).

This classical strategy is ω_1^{ck} -computable, using the well-ordering of the reals from the previous slide.

The Borel Dual Ramsey Theorem

Borel Dual Ramsey Theorem (Carlson & Simpson 1984) For any $k, \ell < \omega$, suppose we ℓ -color all the *k*-partitions of \mathbb{N} . If the coloring is Borel, then there is a partition p of \mathbb{N} into infinitely many pieces such that any coarsening of p down to k pieces has the same color.

Theorem. (TWW) The Borel Dual Ramsey Theorem fails in *HYP*, even for 2-partitions.

Proof. It is well-known that the Dual Ramsey Theorem fails without some niceness condition on the coloring. A standard construction is:

- Let $(p_{\alpha})_{\alpha < \mathfrak{c}}$ be a well-ordering of the infinite partitions.
- At stage α, consider all coarsenings of p_α down to exactly 2 pieces. There are continuum many such 2-partitions, but less than continuum-many 2-partitions have been colored so far. Pick two not yet colored, and color them opposite colors.

This construction can be implemented in an ω_1^{ck} -computable way in HYP.

Descriptive combinatorics - graph coloring

Theorem. (Marks '16) For every $n \ge 2$ there is a Borel *n*-regular acyclic graph with no Borel *n*-coloring.

Proposition. (Towsner, Weisshaar, W.) In *HYP*, every *n*-regular Borel acyclic graph has a Borel 2-coloring.

Proof. The classical choice-based construction of a 2-coloring can be implemented on any *n*-regular graph in an ω_1^{ck} -computable way, using *n*-regularity and admissibility of ω_1^{ck} to identify each connected component by some "finite" computation stage.

Theorem. (Conley, Marks, Tucker-Drob '16) For all $n \ge 3$, any Borel *n*-regular acyclic graph has a measurable *n*-coloring and an *n*-coloring with the property of Baire.

Question. Is Marks' theorem above provable in CD-PB or CD-M?

Descriptive combinatorics - perfect matching

- Using AC, any *n*-regular bipartite graph has a perfect matching.
- However, Borel perfect matchings may not exist, even for *n*-regular Borel bipartite graphs.

Proposition. (Towsner, Weisshaar, W.) In *HYP*, every *n*-regular CD-Borel bipartite graph has a CD-Borel perfect matching.

Proof. The classical choice-based construction of a perfect matching can be implemented on any *n*-regular graph in an ω_1^{ck} -computable way, using *n*-regularity and admissibility of ω_1^{ck} to identify each connected component by some "finite" computation stage.

Compare:

Theorem. (Marks '16) There is a 3-regular Borel bipartite graph with no Borel perfect matching.

Theorem. (Kun '21) There is a 3-regular acyclic Borel bipartite graph with no measurable perfect matching.

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Theorem (TWW) In *HYP*, there is a Borel graph such that every vertex has degree at most 2, but this graph has no Borel 2-coloring.

Proof. We can ω_1^{ck} -compute a graph which diagonalizes against all ω_1^{ck} -computable colorings. To defeat the eth ω_1^{ck} -computable coloring, set out two vertices which are connected to nothing. If the eth algorithm colors them both, connect them in an even or odd length chain to make the coloring wrong.

Method of decorating trees

Setup: Suppose ${\mathcal M}$ an $\omega\text{-model}$ that is hyperarithmetically closed and has pseudo-ordinals.

Suppose P_{α}, N_{α} are sets of Borel rank $\sim \alpha$ that are pairwise disjoint and

$$\mathcal{M} \subseteq igcup_{lpha \in \mathsf{Ord} \cap M} \mathcal{P}_{lpha} \cup \mathcal{N}_{lpha}$$

For example, if A is $\Delta_1(L_{\omega_1^{ck}})$, then we could have

- $P_{\alpha} = \{X : X \in A \text{ and this is first witnessed by } L_{\alpha}\}$
- $N_{\alpha} = \{X : X \notin A \text{ and this is first witnessed by } L_{\alpha}\}$

Then in ${\mathcal M}$ there is a completely determined Borel code for

$$\mathcal{M} \cap \bigcup_{\alpha \in \mathsf{Ord} \cap \mathcal{M}} P_{\alpha}$$

Now, starting with an ill-founded tree T of rank α^* , for all $\alpha < \alpha^*$ we will decorate it with Borel codes for P_{α} and N_{α} as follows:



Only add P_{α} and N_{α} to nodes of rank larger than these decorations. In this way the rank of T is not increased.

Computing evaluation maps



This decorated T is completely determined on elements $X \in P_{\alpha} \cup N_{\alpha}$. The evaluation map f can be computed in about α jumps of X as follows.

- On nodes of rank $<\approx \alpha$, use $X^{(\alpha)}$ directly to compute f
- On nodes of rank $\geq \approx \alpha$, f is constant 0 or 1 depending on if X is in P_{α} or N_{α} .

- **Problem:** If we decorate with P_1 and then with P_{α} , we lost the benefit of decorating with P_1
- Solution: Also decorate the decorations.
- This results in a tree T which \mathcal{M} believes is a CD-Borel code for $\bigcup_{\alpha} P_{\alpha}$.

- Is there a Borel combinatorial zoo below ATR₀? Details?
- Are there any theorems of ordinary math or Borel combinatorics equivalent to CD-PB or CD-M?
- Is there another regularity property of Borel sets which suffices to ensure those theorems about Borel sets which hold by either measure or category arguments?
- What is the reverse math strength of "There is a Borel *d*-regular acyclic graph with no Borel *d*-coloring" for *d* ≥ 3?

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