TORSION-FREE ABELIAN GROUPS ARE BOREL COMPLETE

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Abstract. We prove that the Borel space of torsion-free Abelian groups with domain ω is Borel complete, i.e., the isomorphism relation on this Borel space is as complicated as possible, as an isomorphism relation. This solves a long-standing open problem in descriptive set theory, which dates back to the seminal paper on Borel reducibility of Friedman and Stanley from 1989.

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Part 1: Completeness of TFAB

- Paolini + Shelah, Torsion-free abelian groups are Borel complete. Archive.

Part 2: Short idea of the proof

Idea of the proof in 4 steps.

Part 3: G_δ-Polish groups

- Paolini + Shelah, On the existence of uncountable Hopfian and co-Hopfian A.B. Archive.

- Paolini + Shelah, G_δ-Polish groups are complete co-analytic. In preparation.

Part 1: Completeness of TFAB

Seminal paper on DST of ctbl. structures

[3] H. Friedman and L. Stanley. *A Borel reducibility theory for classes of countable structures.* J. Symb. Log. **54** (1989), no. 03, 894-914.

~~possible, as an isomorphism relation~~. The Borel completeness of countable abelian group theory is particularly interesting from the perspective of model theory, as this class is model theoretically "low", i.e stable (in the terminology of [10]). In fact, as already observed in [3], Borel reducibility can be thought of as a weak version of $\mathfrak{L}_{\omega_1,\omega}$-interpretability, and for other classes of countable structures such as groups or fields much stronger results than Borel completeness exist, as in such cases we can first-order interpret graph theory, but such classes are unstable, while abelian group theory is stable. ~~Reference [8] starts a systematic study of the relations between~~

Fact 1.1. *The set K^L_ω of structures with domain ω in a given countable language L is endowed with a standard Borel space structure $(\mathrm{K}^L_\omega, \mathcal{B})$. Every Borel subset of this space $(\mathrm{K}^L_\omega, \mathcal{B})$ is naturally endowed with the Borel structure induced by $(\mathrm{K}^L_\omega, \mathcal{B})$.*

For example, if take $L = \{e, \cdot, ()^{-1}\}$, and we let K' to be one of the following:

(a) the set of elements of K^L_ω which are groups;
(b) the set of elements of K^L_ω which are abelian groups;
(c) the set of elements of K^L_ω which are torsion-free abelian groups;
(d) the set of elements of K^L_ω which are n-nilpotent groups, for some $n < \omega$;

then we have that K' is a Borel subset of $(\mathrm{K}^L_\omega, \mathcal{B})$, and so Fact 1.1 applies.

Notation 1.6. *(1) We denote by* Graph *the class of graphs.*
(2) We denote by Gp *the class of groups.*
(3) We denote by AB *the class of abelian groups.*
(4) We denote by TFAB *the class of torsion-free abelian groups.*
(5) Given a class K *we denote by* K_λ *the set of structures in* K *with domain* λ.

(λ a cardinal)

We now introduce two notions of reductions:

1. Reductions between subsets of Borel spaces;

2. Reductions between equivalence relations def. on Borel spaces.

1 of previous slides

Definition 1.2. *Let X_1 and X_2 be two standard Borel spaces, and let also $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$. We say that Y_1 is Borel reducible to Y_2, denoted as $Y_1 \leqslant_R Y_2$, when there is a Borel map $\mathbf{B} : X_1 \to X_2$ such that for every $x \in X_1$ we have:*

$$x \in Y_1 \Leftrightarrow \mathbf{B}(x) \in Y_2.$$

Notice that Definition 1.2 covers in particular the case $X_1 = K' \times K'$ for K' as in Fact 1.1, and so for example Y_1 could be the isomorphism relation on K'. Also,

Definition 1.3. *Let X_1 be a Borel space and $Y_1 \subseteq X_1$. We say that Y_1 is* complete analytic *(resp. complete co-analytic) if for every Borel space X_2 and analytic subset (resp. co-analytic subset) Y_2 of X_2 we have that $Y_2 \leqslant_R Y_1$.*

② of previous slides

Definition 1.4. *Let X_1 and X_2 be two standard Borel spaces, and let also E_1 be an equivalence relation defined on X_1 and E_2 be an equivalence relation defined on X_2. We say that E_1 is Borel reducible to E_2, denoted as $E_1 \leqslant_B E_2$, when there is a Borel map $\mathbf{B} : X_1 \to X_2$ such that for every $x, y \in X_1$ we have:*

$$xE_1y \Leftrightarrow \mathbf{B}(x)E_2\mathbf{B}(y).$$

Remark 1.5. *Notice that in the context of Definitions 1.2 and 1.4, $E_1 \leqslant_R E_2$ and $E_1 \leqslant_B E_2$ have two different meaning, as in the first case the witnessing Borel function has domain $X \times X$, while in the second case it has domain X. Furthermore, notice that $E_1 \leqslant_B E_2$ implies $E_1 \leqslant_R E_2$ (but the converse need not hold, see 1.7).*

Important to avoid confusions!!!

Definition 1.6. *Let K_1 be a Borel class of structures with domain ω and let $\cong_1$ be the isomorphism relation on K_1. We say that K_1 is* Borel complete *(or, in more modern terminology, $\cong_1$ is S_∞-complete) if for every Borel class K_2 of structures with domain ω there is a Borel map $\mathbf{B} : \mathrm{K}_2 \to \mathrm{K}_1$ such that for every $A, B \in \mathrm{K}_2$:*

$$A \cong_2 B \;\Leftrightarrow \mathbf{B}(A) \cong_1 \mathbf{B}(B),$$

that is, the isomorphism relation on the space K_2 is Borel reducible (in the sense of Definition 1.4) to the the isomorphism relation on the space K_1.

Recall: $\cong$ on K_1 is as complicated as possible as an $\cong$ relation on the structs

Fact 1.7 ([3])**.** *Let K be a Borel class of structures with domain ω. If K is Borel complete, then its isomorphism relation is a complete analytic subset of $\mathrm{K} \times \mathrm{K}$, but the converse need not hold, as for example abelian p-groups with domain ω have complete analytic isomorphism relation but they are* not *a Borel complete space.*

Examples

projective planes (Poizat)

(i) countable graphs, linear orders and trees are Borel complete;
(ii) torsion abelian groups have complete analytic $\cong$ but are *not* Borel complete;
(iii) nilpotent groups of class 2 and exponent p (p a prime) are Borel complete[1];
(iv) the isomorphism relation on finite rank torsion-free abelian groups is Borel.

'89 paper

In [3] Friedman and Stanley state explicitly:

> There is, alas, a missing piece to the puzzle, namely our conjecture that torsion-free abelian groups are complete. [...] We have not even been able to show that the isomorphism relation on torsion-free abelian groups is complete analytic, nor, in another direction, that the class of all abelian groups is Borel complete. We consider these problems to be among the most important in the subject.

This problem is the main objective of our talk

A little bit of abelian group theory

<u>Def.</u> $G \in AB$ is torsion-free when $\forall g \in G \setminus \{0\}$ and $0 < n < \omega$ we have that $ng \neq 0$.

<u>Def.</u> $G \in AB$ is rigid when $\mathrm{Aut}(G)$ consists only of id_G and $-\mathrm{id}_G : G \to G$

$$g \mapsto -g$$

Additive notation as we work in AB!

Examples (list) of TFAB:

(i) $\mathbb{Z}$, $\mathbb{Q}$;

Direct sum

(ii) $\forall \alpha \in \mathrm{Ord}$, $\bigoplus_{\beta<\alpha} \mathbb{Z}$ and $\bigoplus_{\beta<\alpha} \mathbb{Q}$;

(iii) Subgroups of $\bigoplus_{\beta<\alpha} \mathbb{Q}$.

<u>Def</u>: $G \in AB$, $g \in G$, $0 < n < \omega$. We say that g is n-divisible if $\exists\, h \in G$ s.t. $nh = g$.

G is divisible if $\forall\, g \in G$ and $0 < n < \omega$

g is n-divisible.

5. REMARKS. Every torsion free divisible group D of rank α is a direct sum of α copies of the additive group of rational numbers, and D contains an isomorphic copy of every torsion free Abelian group of rank $\leq \alpha$

Subgroups of $\bigoplus_{\beta<\alpha} \mathbb{Q}$.

is not an example but a description of the torsion-free AB groups

$\mathbb{Z}$ not divisible	$\mathbb{Q}$ divisible

→ We say that $G \in AB$ is TORSION when $\forall g \in G\ \exists 1 \leq n < \omega$ s.t. $ng = 0$.

→ Given $G \in AB$ and p a prime number:

$$Tor(G) = \{ g \in G \mid \exists 1 \leq n < \omega,\ ng = 0 \}$$

$$Tor_p(G) = \{ g \in G \mid \exists 1 \leq n < \omega,\ p^n g = 0 \}$$

→ $G \in AB$ is a p-group iff $G = Tor_p(G)$.

Classes of groups	Compl. of $\cong$
① Torsion $G \in AB$	NOT Borel complete
② $G \in TFAB$	?
③ Mixed $G \in AB$	?

[Notice that the torsion subgroup $Tor(G)$ need not be a direct summand of G, so ③ does not reduce to ① + ②.]

Problem Is $TFAB_\omega$ Borel complete?

History part Friedman-Stanley '89 paper

The challenge was taken by several mathematicians. The first to work on this problem was Hjorth, which in [6] proved that any Borel isomorphism relation is Borel (*) reducible to the isomorphism relation on countable torsion-free abelian groups, and that in particular the isomorphism relation on TFAB_ω is not Borel, leaving though open the question whether TFAB_ω is a Borel complete class, or even whether the isomorphism relation on TFAB_ω is complete (**) analytic (cf. Fact 1.5).

(*) in the sense of 2 from previous slides

(**) in the sense of 1 " " "

The problem resisted further attempts of the time and the interest moved to another very interesting problem on torsion-free abelian groups: for $1 \leqslant n < m < \omega$, is the isomorphism relation on torsion-free abelian groups of rank n strictly less complex than the isomorphism relation on torsion-free abelian groups of rank m?

Very interesting problem with motivation

Whereas there are fairly large classes of torsion groups whose structures can be described in terms of satisfactory invariants, there are only a very few and rather restricted classes of torsion-free groups for which satisfactory structure theory is known. These include the torsion-free groups of rank 1 and their direct sums, but no other major classes; even for groups of finite rank no useful complete systems of invariants are known. Naturally, one can establish certain schemes for constructing torsion-free groups, which provide a certain amount of information about their structures, but the schemes so far known fail to give an acceptable solution to the basic problem of deciding when two groups given by different schemes are isomorphic.

[6] Laszlo Fuchs. *Infinite abelian groups. Vol. II.* Pure and Applied Mathematics, Vol. 36-II Academic Press, New York-London 1973.

$\cong_n$ is iso relation on $G \in TFAB_\omega$ s.t. $\mathrm{rk}(G) = n$.

$\cong^*_n$ is iso relation on $G \in TFAB_\omega$ s.t. $\begin{cases} \mathrm{rk}(G) = n \\ G \text{ rigid} \end{cases}$

<u>Theorem (Thomas)</u> $\cong_n \, <_B \, \cong_{n+1}$

<u>Theorem (Thomas)</u> $\cong^*_{n+1} \, \not\leq_B \, \cong_n$

[11] S. Thomas. *On the complexity of the classification problem for torsion-free abelian groups of rank two.* Acta Math. **189** (2002), no. 02, 287-305.

[12] S. Thomas. *The classification problem for torsion-free abelian groups of finite rank.* J. Amer. Math. Soc. **16** (2003), no. 01, 233-258.

Ok, but what about $\cong$ on TFAB_ω ???

(= no major progress).

~~most important in the subject” in [3]~~. The problem remained dormant for various years (at the best of our knowledge), until Downey and Montalbán [2] made some important progress showing that the isomorphism relation on countable torsion-free abelian groups is complete analytic, a necessary but not sufficient condition for Borel completeness, as recalled in Fact 1.5. This was of course possible evidence that the isomorphism relation was indeed Borel complete, as conjectured in [3]. Despite this advancement, the problem of Borel completeness of countable torsion-free abelian groups resisted for other 12 years, until this day, when we prove:

Main Theorem. *The space* TFAB_ω *is Borel complete, in fact there exists a continous map* $\mathbf{B} : \mathrm{Graph}_\omega \to \mathrm{TFAB}_\omega$ *such that for every* $H_1, H_2, \in \mathrm{Graph}_\omega$*:*

$$H_1 \cong H_2 \text{ if and only if } \mathbf{B}(H_1) \cong \mathbf{B}(H_2).$$

Part 2 : Idea of the proof

$\boxed{\text{Step 1}}$ Let M be countable random graph

$$G_2(X) = G_2 = \bigoplus \{ \mathbb{Q}_x \mid x \in X \}$$ Maximal divisibility

$$\geq$$

We construct universal $\longrightarrow$ $$G_1(X) = G_1 = G(1, M)$$

$$\geq$$

$$G_0(X) = G_0 = \bigoplus \{ \mathbb{Z}_x \mid x \in X \}$$ Minimal divisibility

X an infinite set "with structure acting on it."

Step 2 For every $U \subseteq M$ infinite subgraph we define subgroup $\boxed{G_{(1,U)}} \leq G_{(1,M)} = G_1$

Step 3 For $U, V \subseteq M$ as in Step 2:

$$U \cong V \iff G_{(1,U)} \cong G_{(1,V)}$$

(as graphs) (as groups)

Step 4

$$\underline{\underline{B}} : \mathrm{Graph}_\omega \to \mathrm{TFAB}_\omega$$

$$H \underset{\text{continuous}}{\mapsto} F[H] \subseteq M \underset{\text{continuous}}{\mapsto} G_{(1, F[H])} \leq G_1$$

With some "Borel magic" we assume that on the right the domain of $G \in \mathrm{TFAB}_\omega$ is an infinite subset of ω instead of ω.

Finite subsets of M

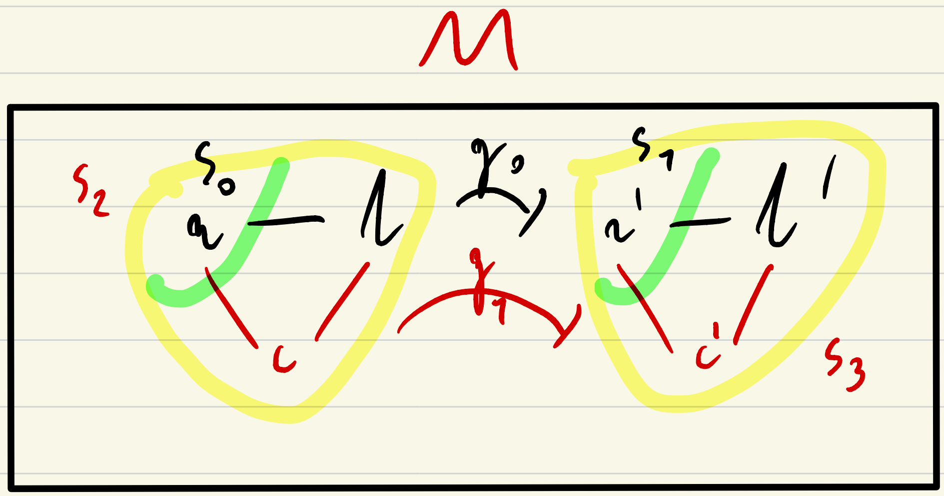
$s_0 = \{a\}$
$s_1 = \{a'\}$

$s_2 = \{a, b, c\}$
$s_3 = \{a', b', c'\}$

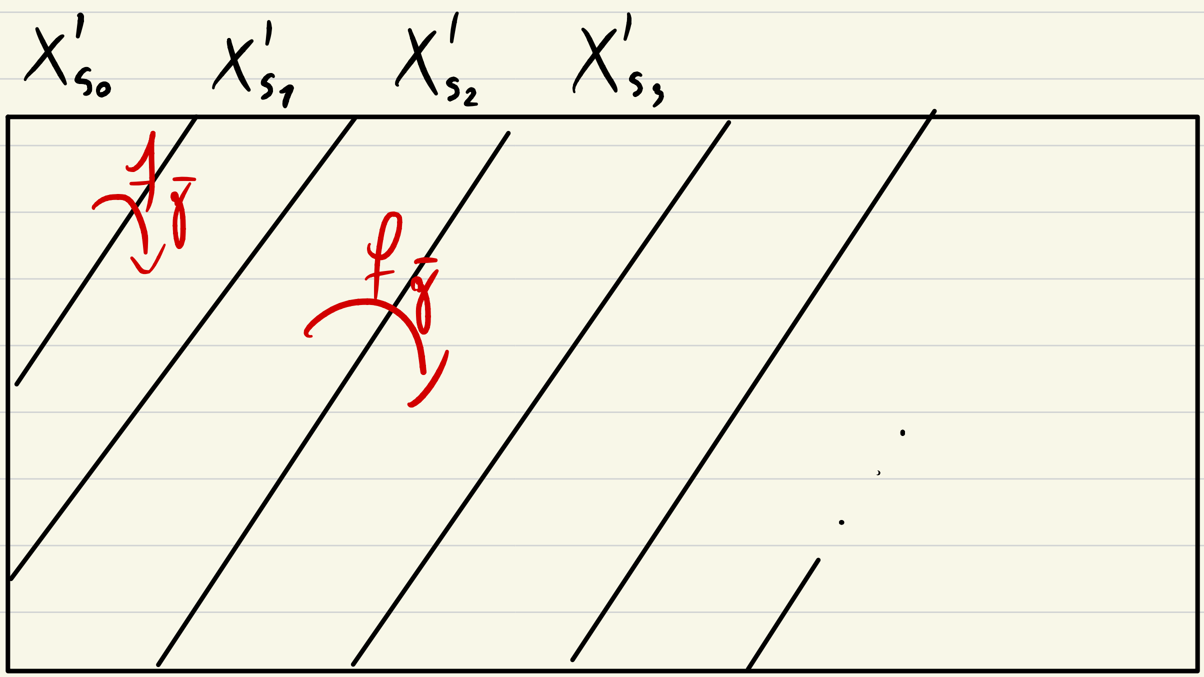
Partial automorph. of M

$g_0 \subseteq g_1$

$\bar{g} = (g_0, g_1)$



The home of the gps. G_ℓ



X

$(X'_s : s \subseteq_\omega M)$ *is a partition of* X *into infinite sets;*

(6) *For* $f_{\bar g} \in \bar f$ *(cf. Definition* 3.3(7)*), let* $\hat f^2_{\bar g}$ *be the unique partial automorphism of* G_2 *which is induced by* $f_{\bar g}$*, explicitly: if* $k < \omega$ *and for every* $\ell < k$ *we have*

that $y^1_\ell \in \mathrm{dom}(f_{\bar g})$, $y^2_\ell = f_{\bar g}(y^1_\ell)$, $q_\ell \in \mathbb{Q}$ *and* $a = \sum_{\ell<k} q_\ell y^1_\ell \in G_2$*, then:*

$$\hat f^2_{\bar g}(a) = \sum_{\ell<k} q_\ell y^2_\ell.$$

Notice that if $\sum_{\ell<k} q_\ell y^1_\ell \in G_1$*, then also* $\sum_{\ell<k} q_\ell y^2_\ell \in G_1$*, by Definition* 3.9(3) *recalling Definition* 3.3(7a) *and* (12a)*, this is relevant for Lemma* 3.10(2)*.*

I know that I did not define C_1 but in order to do it I would need to introduce a lot of technicalities!

Part 3: Co-hybridization groups

Def. Let G be a group. We say that G is co-Hopfian (resp. Hopfian) when every $f \in \mathrm{End}(G)$ which is injective (resp. surjective) is also surjective. (resp injective)

Example The group $\mathbb{Z}$ is Hopfian and not co-Hopfian. Why?

$$f : \mathbb{Z} \to 2\mathbb{Z} \qquad (\text{non-co-Hopfian})$$
$$n \mapsto 2n$$

$$f : \mathbb{Z} \overset{\text{surjective}}{\twoheadrightarrow} \mathbb{Z} \qquad (\text{Hopfian})$$

Then $\ker(f) \leq \mathbb{Z}$ so $\ker(f) = n\mathbb{Z}$ and as $\mathbb{Z}/\ker(f) \cong \mathbb{Z}$ we must have that $n = 0$.

Example The group $\mathbb{Z}(p^\infty)$ (*) is co-Hopfian and not Hopfian. Why? For not-Hopfian

$$\mathbb{Z}(p^\infty) \longrightarrow \mathbb{Z}(p^\infty) \quad \left(\begin{array}{c}\text{surjective}\\ \text{not injective}\end{array}\right)$$

$$\gamma \longmapsto p\gamma$$

(*) Prüfer group: divisible p-group of rank 1

$\mathbb{Z}(p^\infty) = \langle g_1, g_2, \dots \mid p g_1 = 0,\ p g_2 = g_1,\ \dots \rangle$

State of the art on co-Hopfian AB

<u>Fact</u> $G \in \mathrm{TFAB}$ is co-Hopfian iff G is divisible of finite rank, i.e., $\exists\, n < \omega$

$$G \cong \bigoplus_{m<n} \mathbb{Q}.$$

Hence the co-Hopfian $G \in \mathrm{TFAB}_\omega$ are a Borel subset!

Fact [2] Let $G \in AB$ be ω-Hopfian. Then:

$$|\mathrm{Tor}(G)| \leq 2^{\aleph_0}.$$

Fact [2] $\nexists$ $G \in AB$, p-group, ω-Hopfian of size $\aleph_0$.

[2] R. A. Beaumont and R. S. Pierce. *Partly transitive modules and modules with proper isomorphic submodules.* Trans. Amer. Math. Soc. **91** (1959), 209-219.

<u>Question</u> Are there co-Hopfian p-groups in AB of size $\aleph_0 < \lambda \leq 2^{\aleph_0}$???

<u>Fact</u> [4] There are (in ZFC!) groups as ↑ for $\lambda = 2^{\aleph_0}$.

<u>Question</u> What about $\aleph_0 < \lambda < 2^{\aleph_0}$???

This is independent from ZFC [3] !!!

[3] G. Braun and L. Strüngmann. *The independence of the notions of Hopfian and co-Hopfian Abelian p-groups*. Proc. Amer. Math. Soc. 143 (2015), no. 8, 3331-3341.

[4] Peter Crawley *An infinite primary abelian group without proper isomorphic subgroups* . Bull. Amer. Math. Soc. **68** (1962), no. 5, 463-467.

State of the art on Hopfian AB

<u>Fact [24]</u> $\forall \lambda$ $\exists G \in TFAB$ which is endorigid, that is $\forall f \in End(G)$ $\exists m \in \mathbb{Z}$ s.t. $f(x) = mx$ and f is onto iff $m \in \{1, -1\}$.

$$\Downarrow$$

$\forall \lambda$ $\exists G_\lambda \in TFAB_\lambda$ which is Hopfian

$\uparrow$

But this uses STATIONARY SETS, so the construction is non-effective.

What about an EFFECTIVE version???

on a specific notion of "effectiveness" which was suggested for abelian groups by Nadel in [16], i.e., the preservation under any forcing extension of the universe V. We refer to this as the problem of absolute existence (of a group satisfying a certain property). These kind of problems were considered by Fuchs, Göbel, Shelah and others (see e.g. [7, 10, 11]), probably the most important problem in this area is the problem of existence of absolutely indecomposable groups in every cardinality which remains open to this day (despite several partial answers are known).

Problem. *(1) Despite the known necessary restrictions, can we improve (in* ZFC*!) the result from* [2] *that there are no co-Hopfian p-groups of size* $\aleph_0$ *or* $> 2^{\aleph_0}$*?*
(2) Are there co-Hopfian groups in every (resp. arbitrarily large) cardinality?
(3) Are there absolutely Hopfian groups in every cardinality?

Theorem 1.1. *Suppose that* $G \in \mathrm{AB}$ *is reduced and* $\aleph_0 \leqslant |G| < 2^{\aleph_0}$*. If* $\mathfrak{p} > |G|$ *and there is a prime* p *such that* $\mathrm{Tor}_p(G)$ *is infinite, then* G *is not co-Hopfian. In particular there are no infinite reduced* p*-groups* G *of size* $\aleph_0 \leqslant |G| < \mathfrak{p}$*.*

Theorem 1.2. *If* $2^{\aleph_0} < \lambda < \lambda^{\aleph_0}$*, and* $G \in \mathrm{AB}_\lambda$*, then* G *is not co-Hopfian.*

Theorem 1.3. *For all* $\lambda \in \mathrm{Card}$ *there is* $G \in \mathrm{TFAB}_\lambda$ *which is absolutely Hopfian.*

Our theorems (Paolini + Shelah)

From here on: work in preparation !!!

What about countable co-analytic groups?

<u>Question</u> (Thomas) Are the co-analytic groups complete co-analytic in Gr_ω?
(the Borel space of groups with dom ω)

Def. We say that G is nilpotent of class 2 if $\forall\, g, h, k \in G$ we have

$$[[g, h], k] = e,$$

where $[x, y] = x^{-1} y^{-1} x y$ (commutator).

Theorem 1.4. *In* $\mathrm{NipGp}(2)_\omega$ *the set of co-Hopfian groups is complete co-analytic.*

Question 1.6. *Are the co-Hopfian groups complete co-analytic in* AB_ω*?*

We leave this open but we prove:

Theorem 1.5. *If* $G \in \mathrm{AB}_\omega$ *is co-Hopfian and reduced, then for every prime* p *we have that* $\mathrm{Tor}_p(G)$ *is finite and* G *embeds in the profinite group* $\prod_{p \in \mathbb{P}} \mathrm{Tor}_p(G)$.

Reduction to a concrete problem!

Recall: $G \in AB$ is RIGID when:

$$\mathrm{Aut}(G) = \{ id_G, -id_G \}.$$

Theorem 1.3. *The rigid $G \in \mathrm{TFAB}_\omega$ are complete co-analytic in TFAB_ω. In fact, there exists a Borel map $\mathbf{B}$ from Tr_ω to TFAB_ω s.t. for $T \in \mathrm{Tr}_\omega$ we have:*

(1) $\mathbf{B}(T)$ is Hopfian;

(2) if T is well-founded, then $\mathbf{B}(T)$ has only trivial onto endomorphisms;

(3) if T is not well-founded, then $\mathbf{B}(T)$ has a non-trivial (free) automorphism.

$G \in AB$ is boundedly endorigid when

$$\mathrm{End}(G) / B\,\mathrm{End}(G) \cong \mathbb{Z}, \quad \text{where } B\,\mathrm{End}(G)$$

are those $f \in \mathrm{End}(G)$ such that

$\exists\, n < \omega$ such that $n(\mathrm{ran}(f)) = \{0\}$.

$\exists$ of $G \in AB$ as above with

$\mathrm{Tor}(G) = \{ \mathbb{Z}_{p^n} \mid p \text{ prime}, n \geq 1 \}$.

$\Downarrow$

Corollary 1.8. *If $2^{\aleph_0} < \lambda$, then there is a co-Hopfian $G \in \mathrm{AB}$ iff $\lambda = \lambda^{\aleph_0}$.*

THE END

THANK YOU