A dichotomy for σ-Baire class α functions

Andrew Marks (UCLA), joint with Antonio Montalbán (Berkeley)

Work in progress

Dec 8, 2021

- A couple recent open problems in descriptive set theory (e.g. the decomposability conjecture, and the new dichotomy proved in this talk) have required complicated priority arguments to solve.

- A couple recent open problems in descriptive set theory (e.g. the decomposability conjecture, and the new dichotomy proved in this talk) have required complicated priority arguments to solve.
- Some of these types of priority argument proofs can be made simpler using recent ideas from computability theory. In particular we show that Montalbán's recent (2020) "game metatheorem" can be adapted to descriptive set theory. This "topological metatheorem" is about the existence of a strategy in certain game so that result of playing the game for every input yields a continuous function.

Piecewise continuous functions

A function $f\colon X \to Y$ between topological spaces is σ-**continuous** iff there is a countable partition $X = \bigcup_{n\in\mathbb{N}} A_n$ of X so that for every n, $f \restriction A_n$ is continuous.

Piecewise continuous functions

A function $f\colon X \to Y$ between topological spaces is σ-**continuous** iff there is a countable partition $X = \bigcup_{n\in\mathbb{N}} A_n$ of X so that for every n, $f \restriction A_n$ is continuous. The A_n may be arbitrary sets.

Piecewise continuous functions

A function $f\colon X \to Y$ between topological spaces is σ-**continuous** iff there is a countable partition $X = \bigcup_{n\in\mathbb{N}} A_n$ of X so that for every n, $f \restriction A_n$ is continuous. The A_n may be arbitrary sets.

Question (Lusin, 1920s)

Is every Borel function between Polish spaces σ-continuous?

Piecewise continuous functions

A function $f: X \to Y$ between topological spaces is σ-**continuous** iff there is a countable partition $X = \bigcup_{n \in \mathbb{N}} A_n$ of X so that for every n, $f \upharpoonright A_n$ is continuous. The A_n may be arbitrary sets.

Question (Lusin, 1920s)

Is every Borel function between Polish spaces σ-continuous?

No! Consider the Turing jump $j: 2^\omega \to 2^\omega$ where $j(x) = x'$. j is a Borel function and we will show that it is not σ-continuous.

Piecewise continuous functions

A function $f\colon X \to Y$ between topological spaces is σ-**continuous** iff there is a countable partition $X = \bigcup_{n\in\mathbb{N}} A_n$ of X so that for every n, $f \restriction A_n$ is continuous. The A_n may be arbitrary sets.

Question (Lusin, 1920s)

Is every Borel function between Polish spaces σ-continuous?

No! Consider the Turing jump $j\colon 2^\omega \to 2^\omega$ where $j(x) = x'$. j is a Borel function and we will show that it is not σ-continuous.

Assume for a contradiction j is σ-continuous. Then there are partial continuous functions $(g_n)_{n\in\mathbb{N}}$ so that for every $x \in 2^\omega$, $j(x) = g_n(x)$ for some n. Recall a partial function $g\colon 2^\omega \rightharpoonup 2^\omega$ is continuous iff $g(x)$ is uniformly computable from x relative to some oracle $z \in 2^\omega$. Let $z_n \in 2^\omega$ be an oracle relative to which g_n is uniformly computable.

Piecewise continuous functions

A function $f\colon X \to Y$ between topological spaces is σ-**continuous** iff there is a countable partition $X = \bigcup_{n\in\mathbb{N}} A_n$ of X so that for every n, $f \upharpoonright A_n$ is continuous. The A_n may be arbitrary sets.

Question (Lusin, 1920s)

Is every Borel function between Polish spaces σ-continuous?

No! Consider the Turing jump $j\colon 2^\omega \to 2^\omega$ where $j(x) = x'$. j is a Borel function and we will show that it is not σ-continuous.

Assume for a contradiction j is σ-continuous. Then there are partial continuous functions $(g_n)_{n\in\mathbb{N}}$ so that for every $x \in 2^\omega$, $j(x) = g_n(x)$ for some n. Recall a partial function $g\colon 2^\omega \rightharpoonup 2^\omega$ is continuous iff $g(x)$ is uniformly computable from x relative to some oracle $z \in 2^\omega$. Let $z_n \in 2^\omega$ be an oracle relative to which g_n is uniformly computable.

Let $x = \bigoplus_n z_n$. Then for every n, $x \geq_T g_n(x)$. Since $x' = j(x) = g_n(x)$ for some n, we therefore have $x \geq_T x'$. Contradiction! $\square$

Piecewise continuous functions

A function $f\colon X \to Y$ between topological spaces is σ-**continuous** iff there is a countable partition $X = \bigcup_{n\in\mathbb{N}} A_n$ of X so that for every n, $f \restriction A_n$ is continuous. The A_n may be arbitrary sets.

Question (Lusin, 1920s)

Is every Borel function between Polish spaces σ-continuous?

No! Consider the Turing jump $j\colon 2^\omega \to 2^\omega$ where $j(x) = x'$. j is a Borel function and we will show that it is not σ-continuous.

Assume for a contradiction j is σ-continuous. Then there are partial continuous functions $(g_n)_{n\in\mathbb{N}}$ so that for every $x \in 2^\omega$, $j(x) = g_n(x)$ for some n. Recall a partial function $g\colon 2^\omega \rightharpoonup 2^\omega$ is continuous iff $g(x)$ is uniformly computable from x relative to some oracle $z \in 2^\omega$. Let $z_n \in 2^\omega$ be an oracle relative to which g_n is uniformly computable.

Let $x = \bigoplus_n z_n$. Then for every n, $x \geq_T g_n(x)$. Since $x' = j(x) = g_n(x)$ for some n, we therefore have $x \geq_T x'$. Contradiction! $\square$
Historically, the first counterexample was given by Keldiš (1934).

Piecewise continuous functions

A function $f\colon X \to Y$ between topological spaces is σ-**continuous** iff there is a countable partition $X = \bigcup_{n\in\mathbb{N}} A_n$ of X so that for every n, $f \restriction A_n$ is continuous. The A_n may be arbitrary sets.

Question (Lusin, 1920s)

Is every Borel function between Polish spaces σ-continuous?

No! Consider the Turing jump $j\colon 2^\omega \to 2^\omega$ where $j(x) = x'$. j is a Borel function and we will show that it is not σ-continuous.

Assume for a contradiction j is σ-continuous. Then there are partial continuous functions $(g_n)_{n\in\mathbb{N}}$ so that for every $x \in 2^\omega$, $j(x) = g_n(x)$ for some n. Recall a partial function $g\colon 2^\omega \rightharpoonup 2^\omega$ is continuous iff $g(x)$ is uniformly computable from x relative to some oracle $z \in 2^\omega$. Let $z_n \in 2^\omega$ be an oracle relative to which g_n is uniformly computable.

Let $x = \bigoplus_n z_n$. Then for every n, $x \geq_T g_n(x)$. Since $x' = j(x) = g_n(x)$ for some n, we therefore have $x \geq_T x'$. Contradiction! $\square$

Historically, the first counterexample was given by Keldiš (1934).

It is a theorem of Solecki-Zapletal that the Turing jump is a basis for all non σ-continuous Borel functions.

Continuous reducibility (Hertling-Weihrauch '94)

If $f: X \to Y$ and $g: Z \to W$, say that g is **continuously reducible** to f noted $g \leq_c f$ if there are a continuous $\phi: Z \to X$ and partial continuous $\psi: Y \rightharpoonup W$ so that $g = \psi \circ f \circ \phi$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "\psi"] \\ Z \arrow[u, "\phi"] \arrow[r, "g"] & W \end{tikzcd}$$

Continuous reducibility (Hertling-Weihrauch '94)

If $f\colon X \to Y$ and $g\colon Z \to W$, say that g is **continuously reducible** to f noted $g \leq_c f$ if there are a continuous $\phi\colon Z \to X$ and partial continuous $\psi\colon Y \rightharpoonup W$ so that $g = \psi \circ f \circ \phi$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi\uparrow & & \downarrow\psi \\ Z & \xrightarrow{g} & W \end{array}$$

- If $g \leq_c f$ and f is σ-continuous, then g is σ-continuous.

Continuous reducibility (Hertling-Weihrauch '94)

If $f \colon X \to Y$ and $g \colon Z \to W$, say that g is **continuously reducible** to f noted $g \leq_c f$ if there are a continuous $\phi \colon Z \to X$ and partial continuous $\psi \colon Y \rightharpoonup W$ so that $g = \psi \circ f \circ \phi$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "\psi"] \\ Z \arrow[u, "\phi"] \arrow[r, "g"] & W \end{tikzcd}$$

- If $g \leq_c f$ and f is σ-continuous, then g is σ-continuous.
- Any continuous function from $X \to X$ is continuously reducible to the identity function $\mathrm{id} \colon X \to X$.

Continuous reducibility (Hertling-Weihrauch '94)

If $f\colon X \to Y$ and $g\colon Z \to W$, say that g is **continuously reducible** to f noted $g \leq_c f$ if there are a continuous $\phi\colon Z \to X$ and partial continuous $\psi\colon Y \rightharpoonup W$ so that $g = \psi \circ f \circ \phi$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\uparrow & & \downarrow{\scriptstyle\psi} \\ Z & \xrightarrow{g} & W \end{array}$$

- If $g \leq_c f$ and f is σ-continuous, then g is σ-continuous.
- Any continuous function from $X \to X$ is continuously reducible to the identity function $\mathrm{id}\colon X \to X$.
- If $f\colon X \to X$ is a Borel function with uncountable range, then $\mathrm{id}\colon 2^\omega \to 2^\omega$ is continuously reducible to X. (Apply Silver's theorem to the equivalence relation where $x \mathrel{E} y$ if $f(x) = f(y)$)

Continuous reducibility (Hertling-Weihrauch '94)

If $f\colon X \to Y$ and $g\colon Z \to W$, say that g is **continuously reducible** to f noted $g \leq_c f$ if there are a continuous $\phi\colon Z \to X$ and partial continuous $\psi\colon Y \rightharpoonup W$ so that $g = \psi \circ f \circ \phi$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "\psi"] \\ Z \arrow[u, "\phi"] \arrow[r, "g"] & W \end{tikzcd}$$

- If $g \leq_c f$ and f is σ-continuous, then g is σ-continuous.
- Any continuous function from $X \to X$ is continuously reducible to the identity function $\mathrm{id}\colon X \to X$.
- If $f\colon X \to X$ is a Borel function with uncountable range, then $\mathrm{id}\colon 2^\omega \to 2^\omega$ is continuously reducible to X. (Apply Silver's theorem to the equivalence relation where $x \mathrel{E} y$ if $f(x) = f(y)$)
- Any $\mathbf{\Sigma}^0_{n+1}$-measurable function g on 2^ω is continuously reducible to the nth Turing jump $j_n(x) = x^{(n)}$. Let $z \in 2^\omega$ be so that $g(x)$ is uniformly $\Delta^{0,z}_{n+1}$ definable from x. Let $\phi(x) = x \oplus z$ and let $\psi(x)$ witness that $(x \oplus z)^{(n)} \geq_T g(x)$ uniformly.

Continuous reducibility (Hertling-Weihrauch '94)

If $f: X \to Y$ and $g: Z \to W$, say that g is **continuously reducible** to f noted $g \leq_c f$ if there are a continuous $\phi: Z \to X$ and partial continuous $\psi: Y \rightharpoonup W$ so that $g = \psi \circ f \circ \phi$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi \uparrow & & \downarrow \psi \\ Z & \xrightarrow{g} & W \end{array}$$

- If $g \leq_c f$ and f is σ-continuous, then g is σ-continuous.
- Any continuous function from $X \to X$ is continuously reducible to the identity function $\mathrm{id}: X \to X$.
- If $f: X \to X$ is a Borel function with uncountable range, then $\mathrm{id}: 2^\omega \to 2^\omega$ is continuously reducible to X. (Apply Silver's theorem to the equivalence relation where $x \, E \, y$ if $f(x) = f(y)$)
- Any $\mathbf{\Sigma}^0_{n+1}$-measurable function g on 2^ω is continuously reducible to the nth Turing jump $j_n(x) = x^{(n)}$. Let $z \in 2^\omega$ be so that $g(x)$ is uniformly $\Delta^{0,z}_{n+1}$ definable from x. Let $\phi(x) = x \oplus z$ and let $\psi(x)$ witness that $(x \oplus z)^{(n)} \geq_T g(x)$ uniformly.
- Say a function $f: 2^\omega \to 2^\omega$ is **complete $\mathbf{\Sigma}^0_\alpha$-measurable** iff it is $\mathbf{\Sigma}^0_\alpha$-measurable and every $\mathbf{\Sigma}^0_\alpha$ measurable function g is continuously reducible to it.

Continuous reducibility (Hertling-Weihrauch '94)

If $f\colon X \to Y$ and $g\colon Z \to W$, say that g is **continuously reducible** to f noted $g \le_c f$ if there are a continuous $\phi\colon Z \to X$ and partial continuous $\psi\colon Y \rightharpoonup W$ so that $g = \psi \circ f \circ \phi$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\uparrow & & \downarrow{\scriptstyle\psi} \\ Z & \xrightarrow{g} & W \end{array}$$

- If $g \le_c f$ and f is σ-continuous, then g is σ-continuous.
- Any continuous function from $X \to X$ is continuously reducible to the identity function $\mathrm{id}\colon X \to X$.
- If $f\colon X \to X$ is a Borel function with uncountable range, then $\mathrm{id}\colon 2^\omega \to 2^\omega$ is continuously reducible to X. (Apply Silver's theorem to the equivalence relation where $x\,E\,y$ if $f(x) = f(y)$)
- Any $\mathbf{\Sigma}^0_{n+1}$-measurable function g on 2^ω is continuously reducible to the nth Turing jump $j_n(x) = x^{(n)}$. Let $z \in 2^\omega$ be so that $g(x)$ is uniformly $\Delta^{0,z}_{n+1}$ definable from x. Let $\phi(x) = x \oplus z$ and let $\psi(x)$ witness that $(x \oplus z)^{(n)} \ge_T g(x)$ uniformly.
- Say a function $f\colon 2^\omega \to 2^\omega$ is **complete $\mathbf{\Sigma}^0_\alpha$-measurable** iff it is $\mathbf{\Sigma}^0_\alpha$-measurable and every $\mathbf{\Sigma}^0_\alpha$ measurable function g is continuously reducible to it.

The Solecki-Zapletal dichotomy

The Turing jump is the basis for all non σ-continuous functions!

Theorem (Solecki 98, Zapletal 04)

Suppose $f : X \to Y$ is a Borel function between Polish spaces. Then exactly one of the following holds:

- *f is σ-continuous, or*
- *$j \leq_c f$ where j is the Turing jump.*

The Solecki-Zapletal dichotomy

The Turing jump is the basis for all non σ-continuous functions!

Theorem (Solecki 98, Zapletal 04)

Suppose $f : X \to Y$ is a Borel function between Polish spaces. Then exactly one of the following holds:

- *f is σ-continuous, or*
- *$j \leq_c f$ where j is the Turing jump.*

This theorem can be sharpened so that the continuous reduction becomes a topological embedding. The dichotomy in Solecki and Zapletal is stated in terms of the Pawlikowski function, but the Pawlikowski function is continuously equivalent to the Turing jump.

The Solecki-Zapletal dichotomy

The Turing jump is the basis for all non σ-continuous functions!

Theorem (Solecki 98, Zapletal 04)

Suppose $f : X \to Y$ is a Borel function between Polish spaces. Then exactly one of the following holds:

- *f is σ-continuous, or*
- *$j \leq_c f$ where j is the Turing jump.*

This theorem can be sharpened so that the continuous reduction becomes a topological embedding. The dichotomy in Solecki and Zapletal is stated in terms of the Pawlikowski function, but the Pawlikowski function is continuously equivalent to the Turing jump.

Kihara has applied this dichotomy to some problems surrounding Martin's conjecture on Turing invariant functions.

The Solecki-Zapletal dichotomy

The Turing jump is the basis for all non σ-continuous functions!

Theorem (Solecki 98, Zapletal 04)

Suppose $f : X \to Y$ is a Borel function between Polish spaces. Then exactly one of the following holds:

- *f is σ-continuous, or*
- *$j \leq_c f$ where j is the Turing jump.*

This theorem can be sharpened so that the continuous reduction becomes a topological embedding. The dichotomy in Solecki and Zapletal is stated in terms of the Pawlikowski function, but the Pawlikowski function is continuously equivalent to the Turing jump.

Kihara has applied this dichotomy to some problems surrounding Martin's conjecture on Turing invariant functions. Kihara's results were recently generalized by Lutz-Siskind

A proof sketch of Solecki-Zapletal

Assume $f\colon \omega^\omega \to \omega^\omega$. By relativization, we may assume that f is lightface Δ^1_1. By a standard reflection argument $C = \bigcup\{A\colon A \text{ is } \Sigma^1_1 \text{ and } f \restriction A \text{ is continuous}\}$ is a Π^1_1 set. Hence, its complement C^{c} is a Σ^1_1 set and $f \restriction A$ is not continuous for any Σ^1_1 set $A \subseteq C^{\mathsf{c}}$

A proof sketch of Solecki-Zapletal

Assume $f \colon \omega^\omega \to \omega^\omega$. By relativization, we may assume that f is lightface Δ^1_1. By a standard reflection argument $C = \bigcup\{A \colon A \text{ is } \Sigma^1_1 \text{ and } f \restriction A \text{ is continuous}\}$ is a Π^1_1 set. Hence, its complement C^c is a Σ^1_1 set and $f \restriction A$ is not continuous for any Σ^1_1 set $A \subseteq C^\mathsf{c}$

If C^c is empty, then f is σ-continuous.

A proof sketch of Solecki-Zapletal

Assume $f\colon \omega^\omega \to \omega^\omega$. By relativization, we may assume that f is lightface Δ^1_1. By a standard reflection argument $C = \bigcup\{A\colon A \text{ is } \Sigma^1_1 \text{ and } f \restriction A \text{ is continuous}\}$ is a Π^1_1 set. Hence, its complement C^{c} is a Σ^1_1 set and $f \restriction A$ is not continuous for any Σ^1_1 set $A \subseteq C^{\mathsf{c}}$

If C^{c} is empty, then f is σ-continuous.

If C^{c} is nonempty we can perform a finite injury priority construction to continuously reduce the Turing jump to f.

A proof sketch of Solecki-Zapletal

Assume $f\colon \omega^\omega \to \omega^\omega$. By relativization, we may assume that f is lightface Δ^1_1. By a standard reflection argument $C = \bigcup\{A\colon A \text{ is } \Sigma^1_1 \text{ and } f \restriction A \text{ is continuous}\}$ is a Π^1_1 set. Hence, its complement C^c is a Σ^1_1 set and $f \restriction A$ is not continuous for any Σ^1_1 set $A \subseteq C^c$

If C^c is empty, then f is σ-continuous.

If C^c is nonempty we can perform a finite injury priority construction to continuously reduce the Turing jump to f. We'll build ϕ by associating to each string $s \in 2^{<\mathbb{N}}$ a Σ^1_1 set A_s and a basic open set B_s with $A_s \subseteq B_s$, so that the $s \subseteq t$ implies $B_s \supseteq B_t$. We'll then let $\phi(x)$ be the unique y so that $\{y\} = \bigcap_{s \subseteq x} B_s$,

A proof sketch of Solecki-Zapletal

Assume $f\colon \omega^\omega \to \omega^\omega$. By relativization, we may assume that f is lightface Δ^1_1. By a standard reflection argument $C = \bigcup\{A\colon A \text{ is } \Sigma^1_1 \text{ and } f \restriction A \text{ is continuous}\}$ is a Π^1_1 set. Hence, its complement C^c is a Σ^1_1 set and $f \restriction A$ is not continuous for any Σ^1_1 set $A \subseteq C^c$

If C^c is empty, then f is σ-continuous.

If C^c is nonempty we can perform a finite injury priority construction to continuously reduce the Turing jump to f. We'll build ϕ by associating to each string $s \in 2^{<\mathbb{N}}$ a Σ^1_1 set A_s and a basic open set B_s with $A_s \subseteq B_s$, so that the $s \subseteq t$ implies $B_s \supseteq B_t$. We'll then let $\phi(x)$ be the unique y so that $\{y\} = \bigcap_{s\subseteq x} B_s$,

The key idea is that if $f \restriction A$ is not continuous, we can find a basic open neighborhood N_t so that $(f \restriction A)^{-1}(N_t)$ is not relatively open in A. Let $A^* = (f \restriction A)^{-1}(N_t) \setminus \{N_r\colon f(A \cap N_r) \subseteq N_t\}$. Suppose we are building some $\phi(x) \in 2^\omega$ to be inside A^* so that $f(\phi(x)) \subseteq N_t$. The point is that even if we have committed to $\phi(x) \subseteq N_r$ for an arbitrarily long r, since $N_r \cap (f \restriction A)^{-1}(2^{\mathbb{N}} \setminus N_t)$ is nonempty we can at any point switch to building an element of this set instead so that $\phi(x) \subseteq 2^{\mathbb{N}} \setminus N_t$.

$\square$

Generalizing the Solecki-Zapletal dichotomy

A function is Baire class 0 iff it is continuous and f is Baire class α if f is the pointwise limit of a sequence of functions f_n where each f_n is Baire class α_n for some $\alpha_n < \alpha$. In terms of the Borel hierarchy, f is Baire class α if the preimage of every open set is $\mathbf{\Sigma}^0_{\alpha+1}$.

Generalizing the Solecki-Zapletal dichotomy

A function is Baire class 0 iff it is continuous and f is Baire class α if f is the pointwise limit of a sequence of functions f_n where each f_n is Baire class α_n for some $\alpha_n < \alpha$. In terms of the Borel hierarchy, f is Baire class α if the preimage of every open set is $\mathbf{\Sigma}^0_{\alpha+1}$.

Recall for every α there is a complete Baire class α function under continuous reducibility, and any complete Baire class $\alpha + 1$ function is not σ-Baire class α (a countable union of Baire class α partial functions).

Theorem (M.-Montalbán)

Suppose $f : X \to Y$ is a Borel function between Polish spaces. Then exactly one of the following holds:

- *f is σ-Baire class α, or*
- *There is a continuous reduction of a complete Baire class $\alpha + 1$ function to f.*

Generalizing the Solecki-Zapletal dichotomy

A function is Baire class 0 iff it is continuous and f is Baire class α if f is the pointwise limit of a sequence of functions f_n where each f_n is Baire class α_n for some $\alpha_n < \alpha$. In terms of the Borel hierarchy, f is Baire class α if the preimage of every open set is $\boldsymbol{\Sigma}^0_{\alpha+1}$.

Recall for every α there is a complete Baire class α function under continuous reducibility, and any complete Baire class $\alpha+1$ function is not σ-Baire class α (a countable union of Baire class α partial functions).

Theorem (M.-Montalbán)

Suppose $f: X \to Y$ is a Borel function between Polish spaces. Then exactly one of the following holds:

- *f is σ-Baire class α, or*
- *There is a continuous reduction of a complete Baire class $\alpha+1$ function to f.*

The proof is similar in spirit to the $\alpha = 0$ (Solecki-Zapletal) case, except it uses a $0^{(\alpha+1)}$-injury priority argument.

Priority arguments in descriptive set theory

Recently the field of descriptive set theory has been making some serious use of priority arguments. Computability theory uses priority arguments to construct simple objects (e.g. c.e. sets) with complicated properties (e.g. controlling what they can compute). Likewise the uses of priority arguments in descriptive set theory build simple objects (e.g. continuous functions) with complicated properties (e.g. being a Wadge reduction between two complicated sets).

Priority arguments in descriptive set theory

Recently the field of descriptive set theory has been making some serious use of priority arguments. Computability theory uses priority arguments to construct simple objects (e.g. c.e. sets) with complicated properties (e.g. controlling what they can compute). Likewise the uses of priority arguments in descriptive set theory build simple objects (e.g. continuous functions) with complicated properties (e.g. being a Wadge reduction between two complicated sets).

These new uses are more than simply relativizing constructions from computability.

Priority arguments in descriptive set theory

Recently the field of descriptive set theory has been making some serious use of priority arguments. Computability theory uses priority arguments to construct simple objects (e.g. c.e. sets) with complicated properties (e.g. controlling what they can compute). Likewise the uses of priority arguments in descriptive set theory build simple objects (e.g. continuous functions) with complicated properties (e.g. being a Wadge reduction between two complicated sets).

These new uses are more than simply relativizing constructions from computability. These arguments often construct a tree of "approximations" like in the Solecki-Zapletal argument and along each branch a priority argument is taking place. However, additional complexity comes from how the outcomes along different branches now interact with each other (the construction is more "global"). These constructions also use deep results of effective descriptive set theory (e.g. Louveau's analysis of $\Sigma^1_1 \cap \mathbf{\Pi}^0_\alpha$) to find the correct types of approximations that can appropriately respond to the flow of the construction.

Frameworks for priority arguments

The complexity of a priority argument is measured by how complicated it is to compute how the requirements are satisfied.

Frameworks for priority arguments

The complexity of a priority argument is measured by how complicated it is to compute how the requirements are satisfied. As this complexity grows, these constructions become quite complex. Methods have been created for handling the level-by-level combinatorics of these constructions in a more simple and uniform way (e.g. Harrington's worker method, Lempp and Lerman's trees of strategies, Ash's metatheorem, Ash-Knight's η-system, and Montalbán's true stages) .

Frameworks for priority arguments

The complexity of a priority argument is measured by how complicated it is to compute how the requirements are satisfied. As this complexity grows, these constructions become quite complex. Methods have been created for handling the level-by-level combinatorics of these constructions in a more simple and uniform way (e.g. Harrington's worker method, Lempp and Lerman's trees of strategies, Ash's metatheorem, Ash-Knight's η-system, and Montalbán's true stages) .

Recently, Day, Greenberg, and Turetsky have discovered that Montalbán's true stages framework has also arisen independently outside of computability theory: an essentially equivalent framework was used by Louveau and Saint-Raymond in set theory for proving Borel Wadge determinacy in second order arithmetic in 1988.

Frameworks for priority arguments

The complexity of a priority argument is measured by how complicated it is to compute how the requirements are satisfied. As this complexity grows, these constructions become quite complex. Methods have been created for handling the level-by-level combinatorics of these constructions in a more simple and uniform way (e.g. Harrington's worker method, Lempp and Lerman's trees of strategies, Ash's metatheorem, Ash-Knight's η-system, and Montalbán's true stages) .

Recently, Day, Greenberg, and Turetsky have discovered that Montalbán's true stages framework has also arisen independently outside of computability theory: an essentially equivalent framework was used by Louveau and Saint-Raymond in set theory for proving Borel Wadge determinacy in second order arithmetic in 1988. This independent development of the same machinery is compelling evidence that true stages are a natural tool.

Frameworks for priority arguments

The complexity of a priority argument is measured by how complicated it is to compute how the requirements are satisfied. As this complexity grows, these constructions become quite complex. Methods have been created for handling the level-by-level combinatorics of these constructions in a more simple and uniform way (e.g. Harrington's worker method, Lempp and Lerman's trees of strategies, Ash's metatheorem, Ash-Knight's η-system, and Montalbán's true stages) .

Recently, Day, Greenberg, and Turetsky have discovered that Montalbán's true stages framework has also arisen independently outside of computability theory: an essentially equivalent framework was used by Louveau and Saint-Raymond in set theory for proving Borel Wadge determinacy in second order arithmetic in 1988. This independent development of the same machinery is compelling evidence that true stages are a natural tool.

We adapt a new tool for conducting priority arguments: we prove a topological adaptation of Montalbán's recent game metatheorem.

Choquet games

Suppose $\mathbb{A} \subseteq \mathcal{P}(X)$. The **strong Choquet game** $G_{\mathbb{A}}$ is the two player game where the players alternate playing sets $A_i, B_i \in \mathbb{A}$:

$$\begin{array}{llllll} \mathrm{I} & x_0, A_0 & & x_1, A_1 & & \dots \\ \mathrm{II} & & B_0 & & B_1 & \end{array}$$

where $A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \dots$ and $x_i \in B_i$. We say that player II wins the game if $\bigcap_i A_i = \bigcap_i B_i \neq \emptyset$.

Choquet games

Suppose $\mathbb{A} \subseteq \mathcal{P}(X)$. The **strong Choquet game** $G_{\mathbb{A}}$ is the two player game where the players alternate playing sets $A_i, B_i \in \mathbb{A}$:

$$\begin{array}{llllll} \mathrm{I} & x_0, A_0 & & x_1, A_1 & & \dots \\ \mathrm{II} & & B_0 & & B_1 & \end{array}$$

where $A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \dots$ and $x_i \in B_i$. We say that player II wins the game if $\bigcap_i A_i = \bigcap_i B_i \neq \emptyset$.

$\mathbb{A}$ is **strong Choquet** if player II has a winning strategy in $G_{\mathbb{A}}$.

Choquet games

Suppose $\mathbb{A} \subseteq \mathcal{P}(X)$. The **strong Choquet game** $G_{\mathbb{A}}$ is the two player game where the players alternate playing sets $A_i, B_i \in \mathbb{A}$:

$$\begin{array}{lccccc} \text{I} & x_0, A_0 & & x_1, A_1 & & \dots \\ \text{II} & & B_0 & & B_1 & \end{array}$$

where $A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \dots$ and $x_i \in B_i$. We say that player II wins the game if $\bigcap_i A_i = \bigcap_i B_i \neq \emptyset$.

$\mathbb{A}$ is **strong Choquet** if player II has a winning strategy in $G_{\mathbb{A}}$.

Examples of $\mathbb{A}$ that are strong Choquet

- The open sets in a Polish space.

Choquet games

Suppose $\mathbb{A} \subseteq \mathcal{P}(X)$. The **strong Choquet game** $G_{\mathbb{A}}$ is the two player game where the players alternate playing sets $A_i, B_i \in \mathbb{A}$:

$$\begin{array}{llllll} \mathrm{I} & x_0, A_0 & & x_1, A_1 & & \ldots \\ \mathrm{II} & & B_0 & & B_1 & \end{array}$$

where $A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \ldots$ and $x_i \in B_i$. We say that player II wins the game if $\bigcap_i A_i = \bigcap_i B_i \neq \emptyset$.

$\mathbb{A}$ is **strong Choquet** if player II has a winning strategy in $G_{\mathbb{A}}$.

Examples of $\mathbb{A}$ that are strong Choquet

- The open sets in a Polish space.
- The Π^0_α subsets of ω^ω.

Choquet games

Suppose $\mathbb{A} \subseteq \mathcal{P}(X)$. The **strong Choquet game** $G_{\mathbb{A}}$ is the two player game where the players alternate playing sets $A_i, B_i \in \mathbb{A}$:

$$\begin{array}{lllllll} \mathrm{I} & x_0, A_0 & & x_1, A_1 & & \dots \\ \mathrm{II} & & B_0 & & B_1 & \end{array}$$

where $A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \dots$ and $x_i \in B_i$. We say that player II wins the game if $\bigcap_i A_i = \bigcap_i B_i \neq \emptyset$.

$\mathbb{A}$ is **strong Choquet** if player II has a winning strategy in $G_{\mathbb{A}}$.

Examples of $\mathbb{A}$ that are strong Choquet

- The open sets in a Polish space.
- The Π^0_α subsets of ω^ω.
- The $\Pi^{0,\mathrm{hyp}}_\alpha$ subsets of ω^ω.

Choquet games

Suppose $\mathbb{A} \subseteq \mathcal{P}(X)$. The **strong Choquet game** $G_{\mathbb{A}}$ is the two player game where the players alternate playing sets $A_i, B_i \in \mathbb{A}$:

$$\begin{array}{lllllll} \mathrm{I} & x_0, A_0 & & x_1, A_1 & & \ldots \\ \mathrm{II} & & B_0 & & B_1 & \end{array}$$

where $A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \ldots$ and $x_i \in B_i$. We say that player II wins the game if $\bigcap_i A_i = \bigcap_i B_i \neq \emptyset$.

$\mathbb{A}$ is **strong Choquet** if player II has a winning strategy in $G_{\mathbb{A}}$.

Examples of $\mathbb{A}$ that are strong Choquet

- The open sets in a Polish space.
- The Π^0_α subsets of ω^ω.
- The $\Pi^{0,\mathrm{hyp}}_\alpha$ subsets of ω^ω.
- The Σ^1_1 subsets of ω^ω (Gandy).

A warning

We will not use any determinacy in this talk!

A warning

We will not use any determinacy in this talk!

The point of Choquet games is to simplify dealing with issues of Baire category, density arguments, etc.

The game $G_{\mathbb{A},\mathbb{B}}$

Given sets $\mathbb{A}, \mathbb{B} \subseteq \mathcal{P}(\omega^\omega)$, consider the game:

$$\begin{array}{cccccccc}
\text{I} & A_0 & & & A_1 & & & \\
\text{oracle} & & n_0 & & & n_1 & & \cdots \\
\text{II} & & & B_0 & & & B_1 &
\end{array}$$

We think of the moves of the oracle as fixed, and not responding to players I and II.
So on turn i,

- Player I plays $A_i \in \mathbb{A}$ where $A_i \subseteq B_{i-1}$
- The oracle plays $n_i \in \mathbb{N}$
- Player II plays $B_i \in \mathbb{B}$ where $B_i \cap A_i \neq \emptyset$ and $B_i \subseteq B_{i-1}$.

Player II wins if $\bigcap_i B_i$ is a singleton.
Call this the game $G_{\mathbb{A},\mathbb{B}}$.

The game $G_{\mathbb{A},\mathbb{B}}$

Given sets $\mathbb{A}, \mathbb{B} \subseteq \mathcal{P}(\omega^\omega)$, consider the game:

$$\begin{array}{lcccccccc}
\text{I} & A_0 & & & A_1 & & & \\
\text{oracle} & & n_0 & & & n_1 & & \cdots \\
\text{II} & & & B_0 & & & B_1 &
\end{array}$$

We think of the moves of the oracle as fixed, and not responding to players I and II.
So on turn i,

- Player I plays $A_i \in \mathbb{A}$ where $A_i \subseteq B_{i-1}$
- The oracle plays $n_i \in \mathbb{N}$
- Player II plays $B_i \in \mathbb{B}$ where $B_i \cap A_i \neq \emptyset$ and $B_i \subseteq B_{i-1}$.

Player II wins if $\bigcap_i B_i$ is a singleton.
Call this the game $G_{\mathbb{A},\mathbb{B}}$.

Winning strategies for II are easy to create assuming $\mathbb{B}$ is strong Choquet.

The game $G_{\mathbb{A},\mathbb{B}}$

Given sets $\mathbb{A}, \mathbb{B} \subseteq \mathcal{P}(\omega^\omega)$, consider the game:

$$\begin{array}{cccccccc}
\text{I} & A_0 & & & A_1 & & & \\
\text{oracle} & & n_0 & & & n_1 & & \cdots \\
\text{II} & & & B_0 & & & B_1 &
\end{array}$$

We think of the moves of the oracle as fixed, and not responding to players I and II.
So on turn i,

- Player I plays $A_i \in \mathbb{A}$ where $A_i \subseteq B_{i-1}$
- The oracle plays $n_i \in \mathbb{N}$
- Player II plays $B_i \in \mathbb{B}$ where $B_i \cap A_i \neq \emptyset$ and $B_i \subseteq B_{i-1}$.

Player II wins if $\bigcap_i B_i$ is a singleton.
Call this the game $G_{\mathbb{A},\mathbb{B}}$.

Winning strategies for II are easy to create assuming $\mathbb{B}$ is strong Choquet.

If σ is a strategy for I, $y \in \omega^\omega$ are the moves of the oracle, and τ is a winning strategy for II, let $\sigma * y * \tau$ be the outcome of the game when it is played using these strategies. That is, the real z where $\{z\} = \bigcap_i B_i$.

Topological adaptation of Montalbán's game metatheorem.

$$
\begin{array}{lcccccccc}
 & \text{I} & A_0 & & & A_1 & & & \\
G_{\mathbb{A},\mathbb{B}}: & \text{oracle} & & n_0 & & & n_1 & & \cdots \\
 & \text{II} & & & B_0 & & & B_1 &
\end{array}
$$

Theorem (M.-Montalbán)

Fix $\alpha < \omega_1$. Let $\mathbb{A}$ be all Σ^1_1 sets and $\mathbb{B}$ be all $\Pi^0_\alpha(\mathsf{HYP})$ sets. There is a complete Baire class α function $T_\alpha : \omega^\omega \to \omega^\omega$ so:
for every strategy σ for I, there is a winning strategy τ for II so $x \mapsto \sigma * T_\alpha(x) * \tau$ is continuous.

Topological adaptation of Montalbán's game metatheorem.

$$
\begin{array}{ccccccccc}
 & \mathrm{I} & A_0 & & & A_1 & & & \\
G_{\mathbb{A},\mathbb{B}}: & \text{oracle} & & n_0 & & & n_1 & & \cdots \\
 & \mathrm{II} & & & B_0 & & & B_1 &
\end{array}
$$

Theorem (M.-Montalbán)

Fix $\alpha < \omega_1$. Let $\mathbb{A}$ be all Σ^1_1 sets and $\mathbb{B}$ be all $\Pi^0_\alpha(\mathrm{HYP})$ sets. There is a complete Baire class α function $T_\alpha\colon \omega^\omega \to \omega^\omega$ so:
for every strategy σ for I, there is a winning strategy τ for II so $x \mapsto \sigma * T_\alpha(x) * \tau$ is continuous.

It is trivial to find a winning strategy for II so that $x \mapsto \sigma * T_\alpha(x) * \tau$ is Baire class α. The magic of the theorem is making this map continuous. $T_\alpha\colon \omega^\omega \to \omega^\omega$ is a "Skolemized" version of an appropriate iterate of the Turing jump. (It is the set of α-true stages of x).

Topological adaptation of Montalbán's game metatheorem.

$$G_{\mathbb{A},\mathbb{B}}: \quad \begin{array}{cccccccc} \mathrm{I} & A_0 & & & A_1 & & & \\ \text{oracle} & & n_0 & & & n_1 & & \cdots \\ \mathrm{II} & & & B_0 & & & B_1 & \end{array}$$

Theorem (M.-Montalbán)

Fix $\alpha < \omega_1$. Let $\mathbb{A}$ be all Σ^1_1 sets and $\mathbb{B}$ be all $\Pi^0_\alpha(\mathrm{HYP})$ sets. There is a complete Baire class α function $T_\alpha \colon \omega^\omega \to \omega^\omega$ so:
for every strategy σ for I, there is a winning strategy τ for II so $x \mapsto \sigma * T_\alpha(x) * \tau$ is continuous.

It is trivial to find a winning strategy for II so that $x \mapsto \sigma * T_\alpha(x) * \tau$ is Baire class α. The magic of the theorem is making this map continuous. $T_\alpha \colon \omega^\omega \to \omega^\omega$ is a "Skolemized" version of an appropriate iterate of the Turing jump. (It is the set of α-true stages of x).

Proving this theorem requires adapting Montalbán's machinery to the setting of descriptive set theory, and also Louveau's theorems on $\Sigma^1_1 \cap \mathbf{\Pi}^0_\alpha$.

Topological adaptation of Montalbán's game metatheorem.

$$\begin{array}{lccccccc} & \text{I} & A_0 & & & A_1 & & \\ G_{\mathbb{A},\mathbb{B}}: & \text{oracle} & & n_0 & & & n_1 & & \cdots \\ & \text{II} & & & B_0 & & & B_1 \end{array}$$

Theorem (M.-Montalbán)

Fix $\alpha < \omega_1$. Let $\mathbb{A}$ be all Σ^1_1 sets and $\mathbb{B}$ be all $\Pi^0_\alpha(\mathrm{HYP})$ sets. There is a complete Baire class α function $T_\alpha \colon \omega^\omega \to \omega^\omega$ so:
for every strategy σ for I, there is a winning strategy τ for II so $x \mapsto \sigma * T_\alpha(x) * \tau$ is continuous.

It is trivial to find a winning strategy for II so that $x \mapsto \sigma * T_\alpha(x) * \tau$ is Baire class α. The magic of the theorem is making this map continuous. $T_\alpha \colon \omega^\omega \to \omega^\omega$ is a "Skolemized" version of an appropriate iterate of the Turing jump. (It is the set of α-true stages of x).

Proving this theorem requires adapting Montalbán's machinery to the setting of descriptive set theory, and also Louveau's theorems on $\Sigma^1_1 \cap \mathbf{\Pi}^0_\alpha$.

More generally, the topological metatheorem is true when $\mathbb{B} \subseteq \mathbb{A} \subseteq \mathcal{P}(X)$ are countable collections of subsets of a Polish space X, $\mathbb{B}$ consists of $\mathbf{\Pi}^0_\alpha$ sets, and $\mathbb{A}, \mathbb{B}$ satisfy certain closure properties.

An example application: Borel Wadge determinacy

The following theorem is a trivial consequence of Borel determinacy, but it was a long open question whether this theorem was provable without the full power of Borel determinacy.

Theorem (Louveau and Saint-Raymond '88 in second order arithmetic)

Suppose X is a Polish space and $C \subseteq X$ is Borel and not $\boldsymbol{\Sigma}^0_{\alpha+1}$. Then a complete $\boldsymbol{\Pi}^0_{\alpha+1}$ set is Wadge reducible to C.

Proof. By relativization, we may assume that C is Δ^1_1. Let

$$A_0 = C \setminus \{B \colon B \text{ is } \Pi^{0,\mathrm{hyp}}_{\alpha} \text{ and } B \subseteq C\}.$$

Note that A_0 is $\boldsymbol{\Sigma}^1_1$. We must have $A_0 \neq \emptyset$ since otherwise B would be $\boldsymbol{\Sigma}^0_{\alpha+1}$.

Let $T_\alpha(x)$ be the set of α-true stages of x. There is an index e for a c.e. set so that $\{x\colon W_e^{T^\alpha(x)}\downarrow\}$ is $\mathbf{\Pi}^0_{\alpha+1}$ complete.

Let $T_\alpha(x)$ be the set of α-true stages of x. There is an index e for a c.e. set so that $\{x\colon W_e^{T^\alpha(x)}\downarrow\}$ is $\mathbf{\Pi}^0_{\alpha+1}$ complete.
We define a strategy for player I in the game metatheorem. On their first turn I plays A_0. We now define what player I plays on subsequent turns of game after the oracle has played an initial segment s of $T_\alpha(x)$.

Let $T_\alpha(x)$ be the set of α-true stages of x. There is an index e for a c.e. set so that $\{x\colon W_e^{T^\alpha(x)}\downarrow\}$ is $\mathbf{\Pi}^0_{\alpha+1}$ complete.
We define a strategy for player I in the game metatheorem. On their first turn I plays A_0. We now define what player I plays on subsequent turns of game after the oracle has played an initial segment s of $T_\alpha(x)$.
If $W_e^s\uparrow$, then on turn n I plays a set $A_n \subseteq B_{n-1} \cap A_{n-1}$, using a winning strategy for II in the Choquet game on Σ^1_1 sets to ensure that if $W_e^{T^\alpha(x)}\uparrow$, then the moves of player I $A_0 \supseteq A_1 \supseteq A_2 \ldots$ have nonempty intersection. Hence, the outcome of the game is in $\bigcap_i A_i$ and hence in A_0 and thus in C.

Let $T_\alpha(x)$ be the set of α-true stages of x. There is an index e for a c.e. set so that $\{x\colon W_e^{T^\alpha(x)}\downarrow\}$ is $\mathbf{\Pi}^0_{\alpha+1}$ complete.
We define a strategy for player I in the game metatheorem. On their first turn I plays A_0. We now define what player I plays on subsequent turns of game after the oracle has played an initial segment s of $T_\alpha(x)$.
If $W_e^s\uparrow$, then on turn n I plays a set $A_n \subseteq B_{n-1} \cap A_{n-1}$, using a winning strategy for II in the Choquet game on Σ^1_1 sets to ensure that if $W_e^{T^\alpha(x)}\uparrow$, then the moves of player I $A_0 \supseteq A_1 \supseteq A_2 \ldots$ have nonempty intersection. Hence, the outcome of the game is in $\bigcap_i A_i$ and hence in A_0 and thus in C.
If $W_e^s\downarrow$, and s is the string of shortest length so this is true, then player I plays the set $A_n = B_{n-1} \setminus C$. This set is nonempty since $B_{n-1} \cap A_{n-1} \neq \emptyset$, $A_{n-1} \subseteq \ldots \subseteq A_0$, and by the definition of A_0. On subsequent moves I plays according to a winning strategy for the Choquet game on Σ^1_1 sets so that $A_n \supseteq A_{n+1} \supseteq \ldots$ have nonempty intersection.

Let $T_\alpha(x)$ be the set of α-true stages of x. There is an index e for a c.e. set so that $\{x\colon W_e^{T^\alpha(x)}\downarrow\}$ is $\mathbf{\Pi}^0_{\alpha+1}$ complete.

We define a strategy for player I in the game metatheorem. On their first turn I plays A_0. We now define what player I plays on subsequent turns of game after the oracle has played an initial segment s of $T_\alpha(x)$.

If $W_e^s\uparrow$, then on turn n I plays a set $A_n \subseteq B_{n-1} \cap A_{n-1}$, using a winning strategy for II in the Choquet game on Σ^1_1 sets to ensure that if $W_e^{T^\alpha(x)}\uparrow$, then the moves of player I $A_0 \supseteq A_1 \supseteq A_2 \dots$ have nonempty intersection. Hence, the outcome of the game is in $\bigcap_i A_i$ and hence in A_0 and thus in C.

If $W_e^s\downarrow$, and s is the string of shortest length so this is true, then player I plays the set $A_n = B_{n-1} \setminus C$. This set is nonempty since $B_{n-1} \cap A_{n-1} \neq \emptyset$, $A_{n-1} \subseteq \dots \subseteq A_0$, and by the definition of A_0. On subsequent moves I plays according to a winning strategy for the Choquet game on Σ^1_1 sets so that $A_n \supseteq A_{n+1} \supseteq \dots$ have nonempty intersection.

By the game metatheorem there is a strategy for the II so the function $f\colon 2^\omega \to X$ sending x to the outcome of this game where I plays the above strategy is continuous. By our definition of the strategy,

$$W_e^{T^\alpha(x)}\downarrow \iff f(x) \in C.$$

Hence, f is a Wadge reduction from a complete $\mathbf{\Pi}^0_{\alpha+1}$ set to C. $\square$

Finishing the dichotomy for Baire class α functions

Our proof of the dichotomy for σ-Baire class α functions is done using Montalbán's full true stages machinery. The true stages machine iteratively defines partial orders $(\leq_\beta)_{\beta\leq\alpha}$ which approximate the truth of $\mathbf{\Sigma}^0_\beta$ sentences by defining trees with unique branches through them.

Finishing the dichotomy for Baire class α functions

Our proof of the dichotomy for σ-Baire class α functions is done using Montalbán's full true stages machinery. The true stages machine iteratively defines partial orders $(\leq_\beta)_{\beta\leq\alpha}$ which approximate the truth of $\mathbf{\Sigma}^0_\beta$ sentences by defining trees with unique branches through them.

We proved the game metatheorem to try and simply the proof of this dichotomy. Our idea was that if f is Δ^1_1 and not σ-Baire class α, let $C = \bigcup\{A\colon A \text{ is } \Sigma^1_1 \text{ and } f \restriction A \text{ is Baire class } \alpha\}$, so C^c is nonempty. Using the same idea as in the proof of Solecki-Zapletal, write down an appropriate strategy for player I in the topological metatheorem. The continuous function given by the metatheorem will be ϕ, and we will be able to construct ψ from the winning strategy for II.

Finishing the dichotomy for Baire class α functions

Our proof of the dichotomy for σ-Baire class α functions is done using Montalbán's full true stages machinery. The true stages machine iteratively defines partial orders $(\leq_\beta)_{\beta\leq\alpha}$ which approximate the truth of $\boldsymbol{\Sigma}^0_\beta$ sentences by defining trees with unique branches through them.

We proved the game metatheorem to try and simply the proof of this dichotomy. Our idea was that if f is Δ^1_1 and not σ-Baire class α, let $C = \bigcup\{A \colon A \text{ is } \Sigma^1_1 \text{ and } f \restriction A \text{ is Baire class } \alpha\}$, so C^c is nonempty. Using the same idea as in the proof of Solecki-Zapletal, write down an appropriate strategy for player I in the topological metatheorem. The continuous function given by the metatheorem will be ϕ, and we will be able to construct ψ from the winning strategy for II.

Unfortunately, we haven't quite gotten this simplification to work; there is a technical issue in adapting the proof of Solecki-Zapletal that is related to how incompatible branches on the tree of approximations to the continuous reduction affect each other.

Finishing the dichotomy for Baire class α functions

Our proof of the dichotomy for σ-Baire class α functions is done using Montalbán's full true stages machinery. The true stages machine iteratively defines partial orders $(\leq_\beta)_{\beta\leq\alpha}$ which approximate the truth of $\mathbf{\Sigma}^0_\beta$ sentences by defining trees with unique branches through them.

We proved the game metatheorem to try and simply the proof of this dichotomy. Our idea was that if f is Δ^1_1 and not σ-Baire class α, let $C = \bigcup\{A\colon A \text{ is } \Sigma^1_1 \text{ and } f \restriction A \text{ is Baire class } \alpha\}$, so C^{c} is nonempty. Using the same idea as in the proof of Solecki-Zapletal, write down an appropriate strategy for player I in the topological metatheorem. The continuous function given by the metatheorem will be ϕ, and we will be able to construct ψ from the winning strategy for II.

Unfortunately, we haven't quite gotten this simplification to work; there is a technical issue in adapting the proof of Solecki-Zapletal that is related to how incompatible branches on the tree of approximations to the continuous reduction affect each other.

So the full true stages proof is currently our best proof of the dichotomy theorem. A hybrid approach which may work is a "tree" game metatheorem that takes place more directly on a tree, instead of having an oracle in the game.

Open problems

- Find more applications of the metatheorem.

Open problems

- Find more applications of the metatheorem.
- (Carroy, 2013) Is continuous reducibility a well-quasi-order on all Borel functions on 2^ω? (Does it have infinite antichains or infinite descending sequences?).

Open problems

- Find more applications of the metatheorem.
- (Carroy, 2013) Is continuous reducibility a well-quasi-order on all Borel functions on 2^ω? (Does it have infinite antichains or infinite descending sequences?).
  If the answer is positive, there is a deep structure inside Borel functions we have been missing which should be useful to understand and analyze them.

Open problems

- Find more applications of the metatheorem.
- (Carroy, 2013) Is continuous reducibility a well-quasi-order on all Borel functions on 2^ω? (Does it have infinite antichains or infinite descending sequences?).
  If the answer is positive, there is a deep structure inside Borel functions we have been missing which should be useful to understand and analyze them.
- Is continuous reducibility on all functions on $2^{\mathbb{N}}$ a well-quasi-order assuming AD^+?

Open problems

- Find more applications of the metatheorem.
- (Carroy, 2013) Is continuous reducibility a well-quasi-order on all Borel functions on 2^ω? (Does it have infinite antichains or infinite descending sequences?).
  If the answer is positive, there is a deep structure inside Borel functions we have been missing which should be useful to understand and analyze them.
- Is continuous reducibility on all functions on $2^{\mathbb{N}}$ a well-quasi-order assuming AD^+?
- Are there versions of Montalbán's game metatheorem beyond the Borel?

Thanks!