UNIFORM AMENABILITY UNIFORM HYPERFINITENESS

Amenability and hyperfiniteness for infinite graphs.

Let G be an infinite, connected graph with vertex degree bound d. What does it mean that G is hyperfinite?

Følner-property seems to be a good candidate.

G is Følner if for any $\epsilon > 0$ there exists some finite subset $F \subset V(G)$ such that

$$\frac{|\partial(F)|}{|F|} \leq \epsilon\,.$$

Consequences of the definition

1. If G is the Cayley graph of a group Γ, then G is Følner if and only if it is amenable.

2. A regular tree can be made Følner by substituting its edges with longer and longer paths very, very sparsely.

The definition should be made more "local".

Let G be **uniformly locally Følner** (ULF) if for every $\epsilon > 0$ there exists some $K > 0$ such that for **any** finite subset H, there exists a subset $F \subset H$ such that the size of F is at most K and

$$\frac{|\partial_H(F)|}{|F|} \leq \epsilon \,,$$

where $\partial_H(F)$ denotes the boundary of F relative to H not the infinite set $V(G)$. (Brodzki, Niblo, Spakula, Willett and Wright, 2013)

Consequences of the new definition

1. All trees, regular or not, are ULF.

2. So, the Cayley graphs of the free groups are ULF. The Cayler graphs of all hyperbolic groups are ULF. The direct product, free product and even extensions of ULF-groups are still ULF. (explained a bit later)

3. There are non-ULF graphs! One consider an expander sequence of finite graphs and connect them to obtain an infinite graph.

There are non-ULF Cayley-graphs!! The Gromov-Osajda non-exact groups.

The Property A of Guoliang Yu.

Let G be an infinite graph. Then, $\mathrm{Prob}(G)$ is the set of all probability measures on the vertices of G. If $f : V(G) \to \mathbb{R}$ and $g : V(G) \to \mathbb{R}$ are two real functions on the vertices then their l_1-distance is defined as $\|f-g\|_1 := \sum_{x\in V(G)} |f(x)-g(x)|$, also $\|f\| := \sum_{x\in V(G)} |f(x)|$.

Definition 1. *The graph G possesses Property A if for any $\epsilon > 0$ there exists $R > 0$ and for each $x \in V(G)$ a probability measure $P(x)$ supported on the R-neighbourhood of x, such that for all adjacent pairs $x, y \in V(G)$*

$$\|P(x) - P(y)\|_1 \leq \epsilon.$$

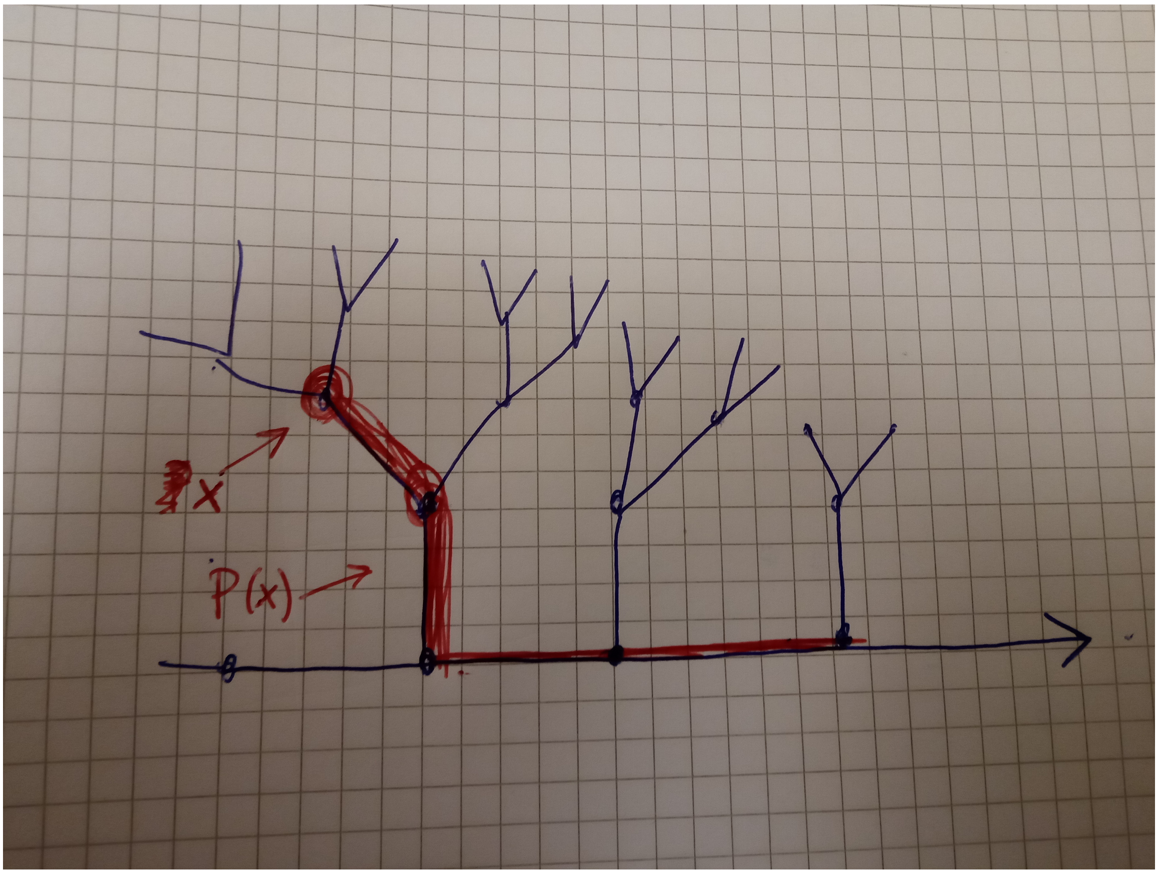

FIGURE 1. The 3-regular tree is of Property A.

(Brodzki et. al.,2013): Property A implies Uniform Local Følner. Conjecture: ULF implies Property A.

(E. 2020): ULF implies Property A.

Idea of the proof: Strong Hyperfiniteness and Weighted Hyperfiniteness.

A K-separator Y in G is a set of vertices such that the complement of Y is a union of subsets of size at most K.

G is **strongly hyperfinite** if for any $\epsilon > 0$ there exists $K > 0$ such that there exists a probability measure on the set of all K-separators in such a way that for any $x \in V(G)$ the probability of K-separators containing x is less than $\epsilon > 0$. (Wrochna, Zivny, Romero 2019 for finite graphs)

G is **weighted hyperfinite** if for any $\epsilon > 0$ there exists $K > 0$ such that for any probability measure w on $V(G)$ one can delete a subset $S \subset V(G)$ of w-measure less than ϵ in such a way that the resulting graph has components of size at most K. (E.-Timár, 2011)

Property A is equivalent to weighted hyperfiniteness. (Sako, 2012)

ULF=> Strong Hyperfiniteness => Weighted hyperfiniteness.

Borel graphs, measured graphs

Let Γ be a finitely generated group with a symmetric generating set Σ and let $\alpha : \Gamma \curvearrowright X$ be a Borel action. The associated **Borel graph** $\alpha_G^{\Gamma,\Sigma}$ is defined in the following way. The vertex set is X and (x, y) is an edge of $\alpha_G^{\Gamma,\Sigma}$ for $x \neq y$ if and only if there is a generator $\sigma \in \Sigma$ such that $y = \alpha(\sigma)(x)$.

Let μ be a probability measure non-singular (quasi-invariant) with respect to the action α and $\mathcal{G}$ be the associated Borel graph. Then, we call the system $(\mathcal{G}, X, \mu)$ a **measured graph**.

Hyperfiniteness:

Let $\mathcal{G}$ be a Borel graph. Let $E_{\mathcal{G}}$ the associated orbit equivalence relation. We call $\mathcal{G}$ hyperfinite, if

$$E = E_1 \subset E_2 \subset \ldots$$

where $\{E_n\}_{n=1}^{\infty}$ are finite equivalence relations.

Amenability:

Let $\mathcal{G}$ be a Borel graph and $E = E_{\mathcal{G}}$. If there exist Borel functions (the Reiter functions) $p_n : E \to [0,1]$ such that

- for any $x \in X$ and $n \geq 1$, $\sum_{z, z \equiv_E x} p_n(x,z) = 1$,
- for any pair $x \equiv_E y$,

$$\lim_{n\to\infty} \sum_{z, z\equiv_E x} |p_n(x,z) - p_n(y,z)| = 0.$$

Proposition: Hyperfiniteness implies amenability.

Conjecture: [Jackson-Kechris-Louveau] Amenability implies Borel-hyperfiniteness.

A measured graph $(\mathcal{G}, X, \mu)$ is μ-amenable if there exists a Borel set $Y \subset X$ of full measure consisting of $\mathcal{G}$-orbits, such that $\mathcal{G}_Y$ is amenable.

A measured graph $(\mathcal{G}, X, \mu)$ is μ-hyperfinite if there exists a Borel set $Y \subset X$ of full measure consisting of $\mathcal{G}$-orbits, such that $\mathcal{G}_Y$ is hyperfinite.

Alternatively, $(\mathcal{G}, X, \mu)$ is μ-hyperfinite if for any $\epsilon > 0$ there exists $K > 0$ such that we have $Z \subset X$, $\mu(Z) < \epsilon$ so that all the components of $\mathcal{G}_{X \setminus Z}$ are of size at most K.

Theorem:[Connes, Feldman, Weiss]
μ-amenability is equivalent to μ-hyperfiniteness.

If Γ is amenable and $\alpha : \Gamma \curvearrowright (X, \mu)$ is a non-singular action, then the associated measured graph is μ-hyperfinite.

On the other hand, any non-amenable group has free, ergodic, non-singular action (necessarily not measure-preserving) such that the associated measured graph is μ-hyperfinite.

Uniform amenability:

(**Anantharaman-Delaroche and Renault**) Let $\mathcal{G}$ be a Borel graph of bounded vertex degrees as above $\epsilon > 0$ and $R \geq 1$. Then, $\mathcal{G}$ is (ϵ, R)-amenable, if there exists a Borel function $p : Y \to \mathrm{Prob}(Y)$ such that

- for all $x \in Y$
$$\mathrm{Supp}(p(x)) \subset B_R(x, \mathcal{G})\,,$$
- and
$$\sum_{x \sim_{\mathcal{G}} y} \|p(x) - p(y)\|_1 \leq \epsilon\,.$$

$\mathcal{G}$ is **Borel-amenable** if for any $\epsilon > 0$ there exists $R \geq 1$ such that $\mathcal{G}$ is (ϵ, R)-amenable.

So, Borel-amenability is much stronger than amenability, it is a sort of uniform amenability.

By definition, all the orbits of $\mathcal{G}$ are of Property A. Basically, Borel-amenability is the dynamical version of Property A. So, if Γ is non-exact then no free action of Γ can be Borel-amenable, but it is well-known that all groups have amenable(hyperfinite) actions.

I do not know what would be the right notion of Borel uniform hyperfiniteness, which might be equivalent to Borel-amenability.

Key definition: The measured graph $(\mathcal{G}, X, \mu)$ is μ-uniformly amenable if there exists an invariant set $Y \subset X$ of full measure such that $\mathcal{G}_Y$ is Borel-amenable.

So, μ-uniform amenability is the measured group theoretic analogue of Property A.

The measured graph $(\mathcal{G}, X, \mu)$ is (ϵ, K)-uniformly hyperfinite, if any subgraph of $\mathcal{G}$ of positive measure is (ϵ, K)-hyperfinite. Again, $(\mathcal{G}, X, \mu)$ is **μ-uniformly hyperfinite** if any for any $\epsilon > 0$ there exists $K > 0$ such that it is (ϵ, K)-uniformly hyperfinite.

It is tempting to conjecture that uniform hyperfiniteness and uniform amenability are equivalent. But...

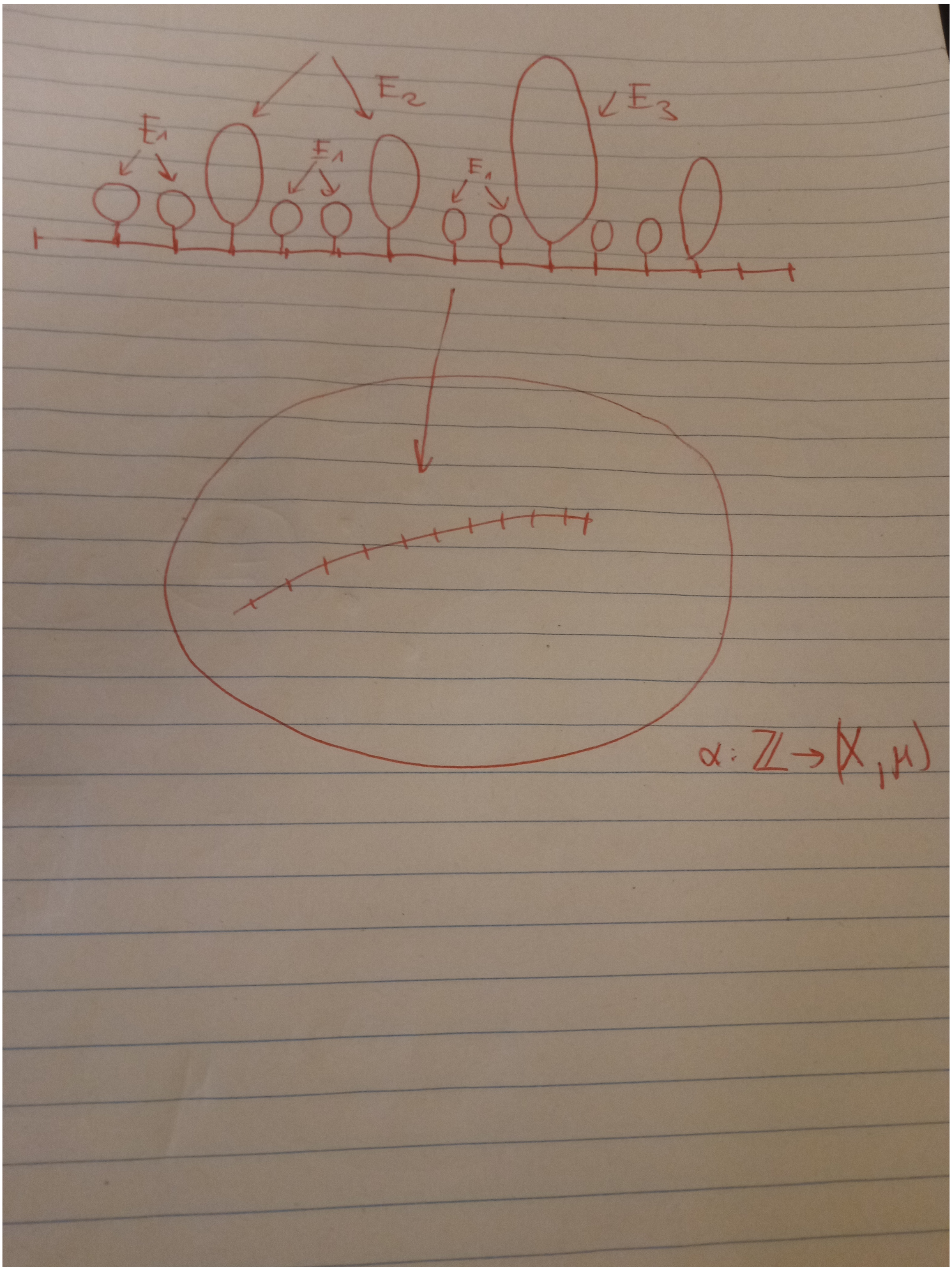

FIGURE 2. Example 1.

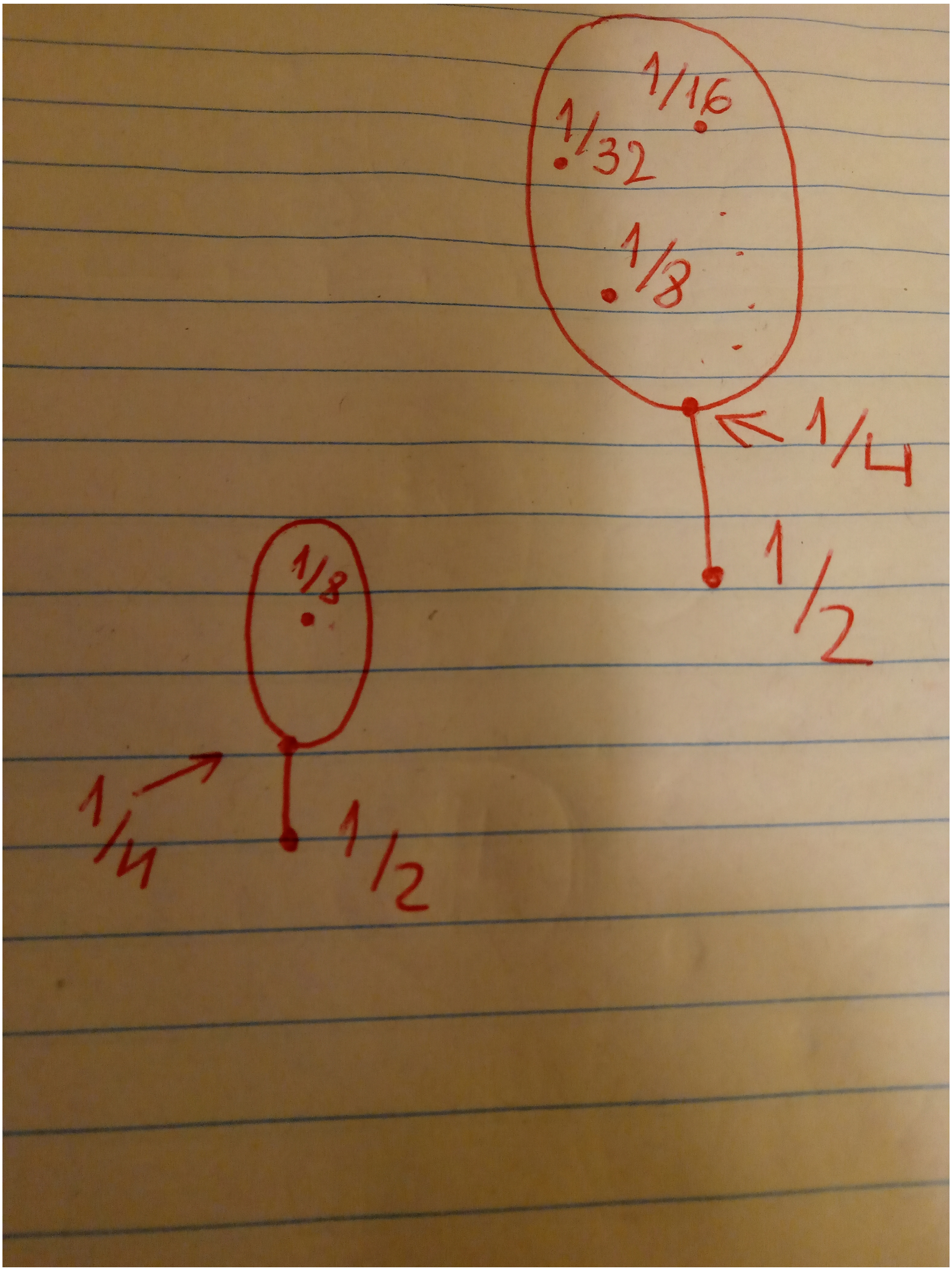

Figure 3. Example 2.

So, there are non-uniformly amenable uniformly hyperfinite measured graphs. But...

Some Radon-Nikodym derivatives in the case of Example 2. are really large.

If $(\mathcal{G}, X, \mu)$ is measured graph, by quasi-invariance we can assume that on all directed edges the Radon-Nikodym derivatives exist (and form a cocycle).

If all the Radon-Nikodym derivatives are bounded by some constant M, then we say that $(\mathcal{G}, X, \mu)$ is of **bounded type**.

(Elek, 2020) For measured graph $(\mathcal{G}, X, \mu)$ of bounded type uniform amenability and uniform hyperfiniteness are equivalent. (AMS Transactions)

Note that there exists topological minimal, non-free actions of the free group with the following property. (Elek-Ceccherini)

1. For an ergodic, invariant measure μ_1 the action is hyperfinite.

2. For an ergodic, invariant measure μ_2 the action is not hyperfinite.

Then μ_1 is not uniformly hyperfinite.

Also, it is known that all groups have free continuous actions with hyperfinite non-singular measures of bounded Radon-Nikodym derivative. For the Gromov-Osajda groups these actions cannot be uniformly hyperfinite.

For exact groups all free hyperfinite actions are non-uniformly hyperfinite.

For non-exact groups all free hyperfinite actions are uniformly hyperfinite (E.- Krolicki)