Logic Seminar (Caltech)

Orbit equivalences of multidimensional Borel flows

Konstantin Slutsky

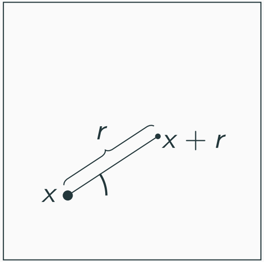
Feb 16th, 2022

Iowa State University

Borel Flows

A multidimensional **Borel flow** is a Borel action $\mathbb{R}^d \curvearrowright \Omega$ on a standard Borel space. The action of $\vec{r} \in \mathbb{R}^d$ upon $x \in \Omega$ will be denoted additively: $x + \vec{r}$.

Irrational rotation on the torus is a classical example of an $\mathbb{R}$ flow.



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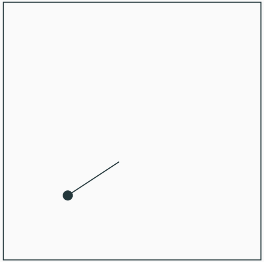

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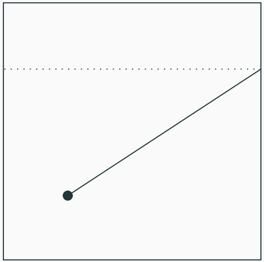

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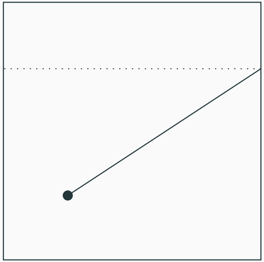
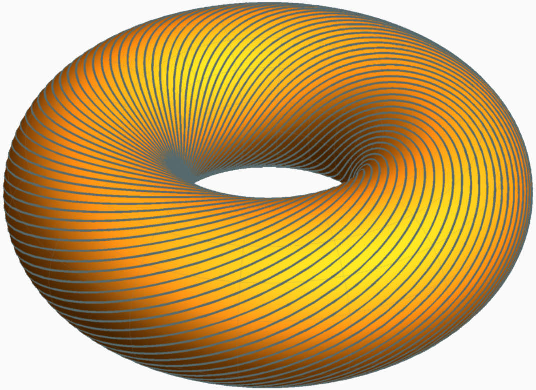


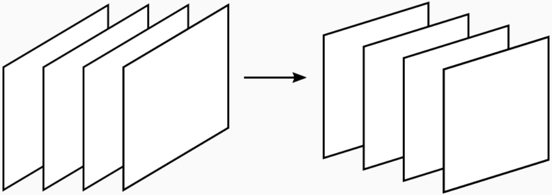
Irrational Rotation on a Torus



Orbit Equivalence

An **orbit equivalence** between flows $\mathbb{R}^d \curvearrowright \Omega_1$ and $\mathbb{R}^d \curvearrowright \Omega_2$ is a Borel bijection $\phi : \Omega_1 \to \Omega_2$ that sends orbits onto orbits:

$$\phi\left(x + \mathbb{R}^d\right) = \phi(x) + \mathbb{R}^d.$$



Strengthening of Orbit Equivalence

When the flow is **free**

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When the flow is **free**

- any orbit of the action can be identified with the affine space $\mathbb{R}^d$;
- one can **transfer any translation-invariant structure** from the Euclidean space onto every orbit of the flow;
- each point considers itself to be the origin, and transfers the structure via the corresponding bijection.

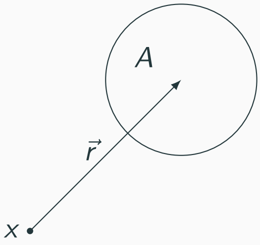
Smooth Structure on Orbits

Transferring topology:

For $x \in \Omega$

$$\mathcal{O}_x = \left\{ A \subseteq x+\mathbb{R}^d : \{\vec{r} \in \mathbb{R}^d \mid x+\vec{r} \in A\} \text{ is open} \right\}.$$

Note that $\mathcal{O}_x = \mathcal{O}_y$ whenever $x + \mathbb{R}^d = y + \mathbb{R}^d$.



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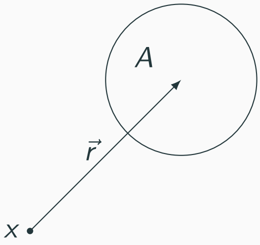

Two flows are **smoothly equivalent** if there exists an orbit equivalence between the phase spaces which is an *orientation preserving diffeomorphism* between orbits.

Smooth Equivalence in Ergodic Theory

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- phase space Ω is endowed with a **probability measure;**
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- all orbit equivalence maps must be **(quasi) measure preserving;**
- and may be defined **up to a null set.**

Smooth Equivalence in Ergodic Theory

Theorem (Feldman–Rudolph, Ornstein–Weiss)

There are continuumly many pairwise time change inequivalent measure preserving ergodic $\mathbb{R}$ *flows.*

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Theorem (Rudolph)

All ergodic measure preserving $\mathbb{R}^d$ flows, $d \geq 2$, are smoothly equivalent.

Theorem (Feldman)

All ergodic **quasi** *measure preserving $\mathbb{R}^d$ flows, $d \geq 2$, are smoothly equivalent.*

Smooth Equivalence in Descriptive Set Theory

Descriptive set theoretical framework is both **more restrictive** (one has to define equivalence on each and every orbit, flows may not preserve any measure) and **more relaxed** (orbit equivalence maps don't need to be measure preserving).

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Recall that a flow $\mathbb{R}^d \curvearrowright \Omega$ is **tame** if it admits a Borel transversal — a Borel set $S \subset \Omega$ that chooses one point from each orbit.

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Theorem (Miller–Rosendal)

All free non tame $\mathbb{R}$ flows are time change equivalent.

Summary of Results on Smooth Equivalence

Dimension	$d = 1$	$d \geq 2$
Ergodic Theory	Many	One
Borel Dynamics	One	

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Theorem (S.)

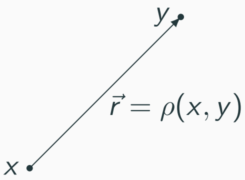
All free non tame $\mathbb{R}^d$ flows, $d \geq 2$, are smoothly equivalent.

Metric on Orbits

Transferring metric:

For $x, y \in \Omega$ within the same orbit there exists a unique $\rho(x, y) \in \mathbb{R}^d$ such that $x + \rho(x, y) = y$.

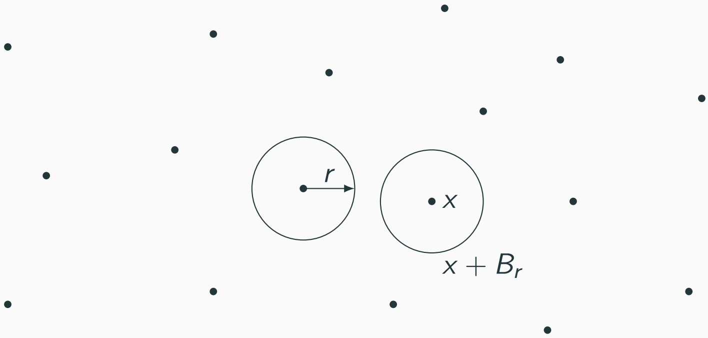
$d(x, y) = ||\rho(x, y)||$



Cross Sections

A **cross section** for an action $\mathbb{R}^d \curvearrowright \Omega$ is a Borel set $\mathcal{C} \subseteq \Omega$ which intersects all orbits of the action and is **lacunary**: for some open ball $B_r \subseteq \mathbb{R}^d$ around the origin

$$(x + B_r) \cap (y + B_r) = \varnothing \text{ whenever } x, y \in \mathcal{C},\ x \neq y.$$



Suspension Flow

With a (bi-infinite, lacunary) cross section $\mathcal{C}$ of an $\mathbb{R}$ flow one associates a **suspension flow**.

Gap function $f_{\mathcal{C}}(x) = \min\{r > 0 : x + r \in \mathcal{C}\}$

Induced automorphism $T : \mathcal{C} \to \mathcal{C}$ given by $T(x) = x + f_{\mathcal{C}}(x)$

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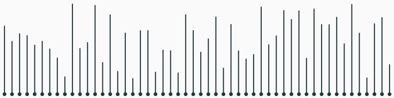

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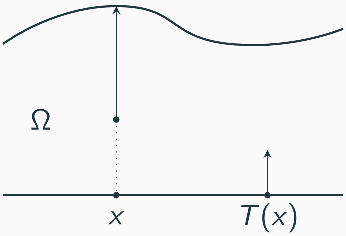
Suspension Flow

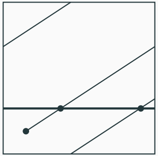
The flow can be modeled on the space Ω **under the graph** of the **gap function**, by flowing upward within a fiber, and then jumping to the next one as determined by T.



$$\Omega = \{(x, t) \in X \times \mathbb{R} : 0 \leq x < f(x)\}$$

Irrational Rotation as a Suspension Flow

The following cross section for the irrational rotation has a constant gap function.



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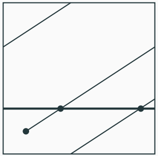


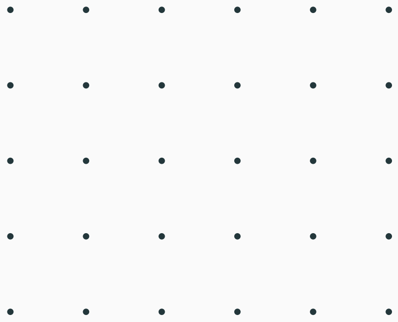
Such a representation is very special — many flows do not admit a cross section with a constant gap function.

Katok's Representation Theorem

Geometrically appealing suspension flow construction is one-dimensional.

An action $\mathbb{Z}^d \curvearrowright X$ can be turned into an $\mathbb{R}^d$ flow on the space $\Omega = X \times [0,1)^d$. For $(x, \vec{s}) \in X \times [0,1)^d$ and $\vec{r} \in \mathbb{R}^d$ let $\vec{n} \in \mathbb{Z}^d$ be such that $\vec{s} + \vec{r} - \vec{n} \in [0,1)^d$; set

$$(x, \vec{s}) + \vec{r} = (x + \vec{n}, \vec{s} + \vec{r} - \vec{n}).$$



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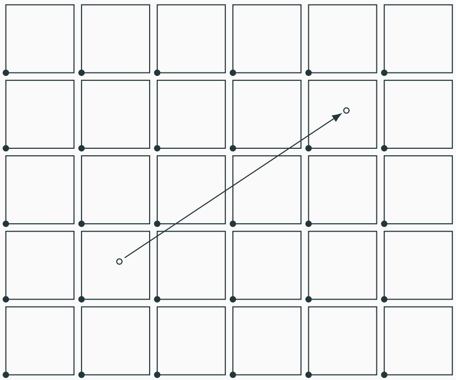

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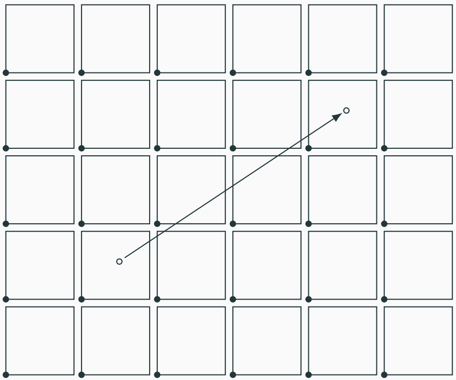
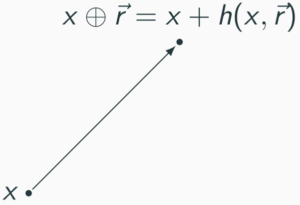
Not every $\mathbb{R}^d$ flow is of this form, just as not every $\mathbb{R}$ flow has cross sections with constant gaps.



Cocycle

Let $x \mapsto x + \vec{r}$ and $x \mapsto x \oplus \vec{r}$ be two flows that have the same orbits. The associated cocycle $h : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is given by the condition

$$x \oplus \vec{r} = x + h(x, \vec{r}) \quad \text{for all } x \in X \text{ and } \vec{r} \in \mathbb{R}^d.$$



Katok's Representation Theorem

Theorem (Katok)

For every free ergodic measure preserving $\mathbb{R}^d$ *flow* $x \mapsto x + \vec{r}$ *and any* $\epsilon > 0$ *there exists a flow* $\mathbb{R}^d \curvearrowright X \times [0,1)^d$ *arising from an ergodic action* $\mathbb{Z}^d \curvearrowright X$ *such that the corresponding cocycle* $h : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ *is* $(1-\epsilon, 1+\epsilon)$*-bi-Lipschitz:*

$$1 - \epsilon \leq \frac{||h(x, \vec{r})||}{||\vec{r}||} \leq 1 + \epsilon.$$

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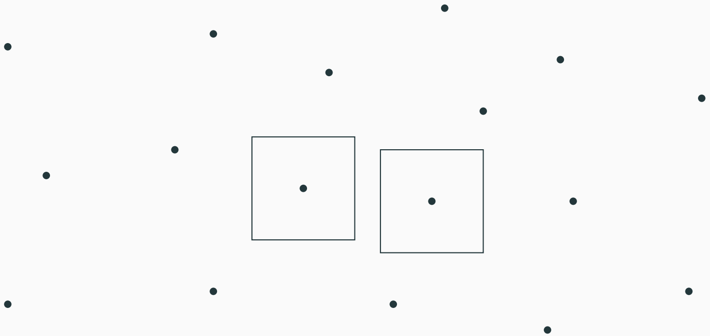
Theorem (S.)

The statement of Katok's Theorem holds within the framework of Borel dynamics.

Layered and Unlayered Toasts

Toasts

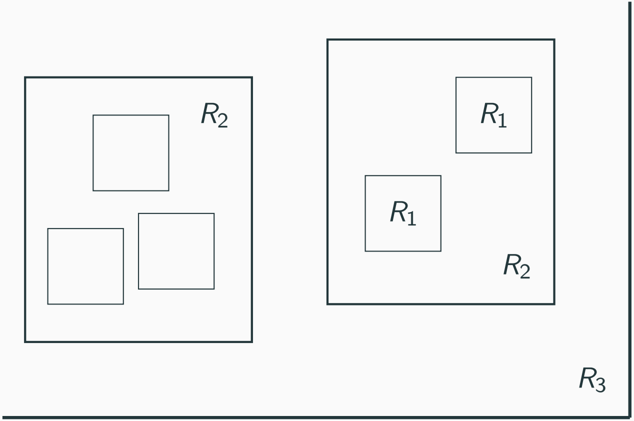
A common pattern is to start with a cross section $\mathcal{C} \subseteq \Omega$ and run a construction within disjoint regions around the points of $\mathcal{C}$.



Layered Toasts in Ergodic Theory

Properties of Regions

- **Exhaustive:** $\bigcup_n R_n = \Omega$.



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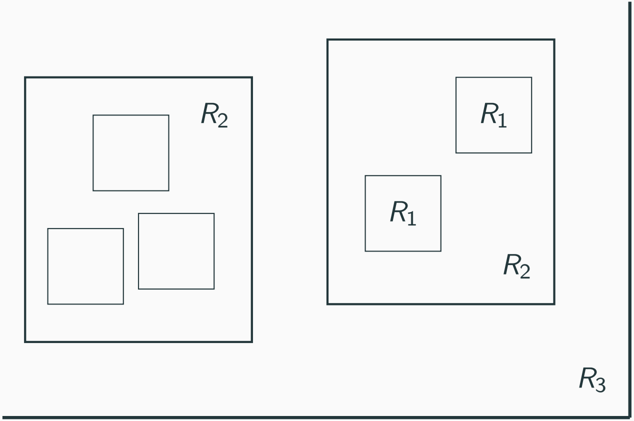

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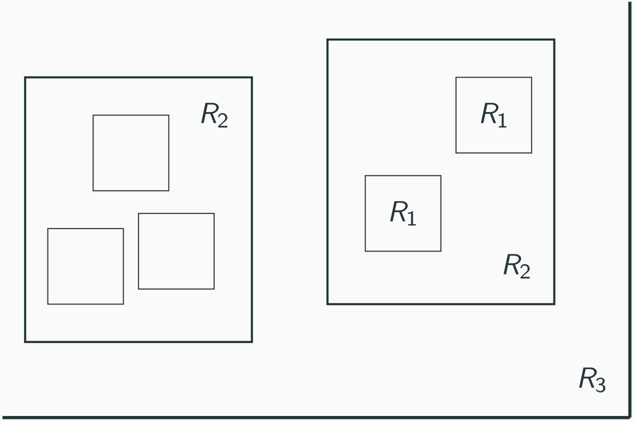
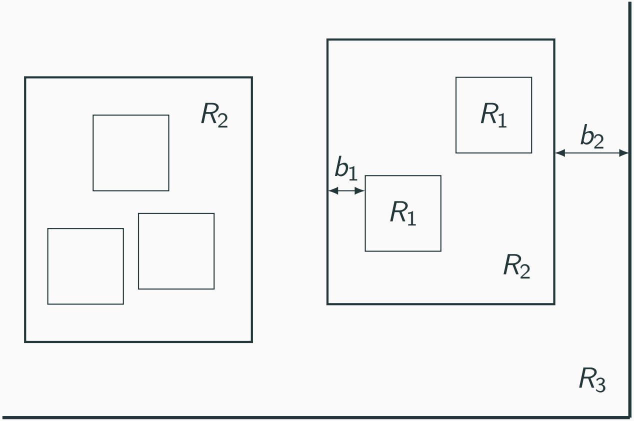
- **Shape:** rectangles.



Layered Toasts in Ergodic Theory

Properties of Regions

- **Exhaustive:** $\bigcup_n R_n = \Omega$.
- **Layered:** $R_n \subseteq R_{n+1}$.
- **Shape:** rectangles.
- **Boundary:** $b_n \to \infty$.



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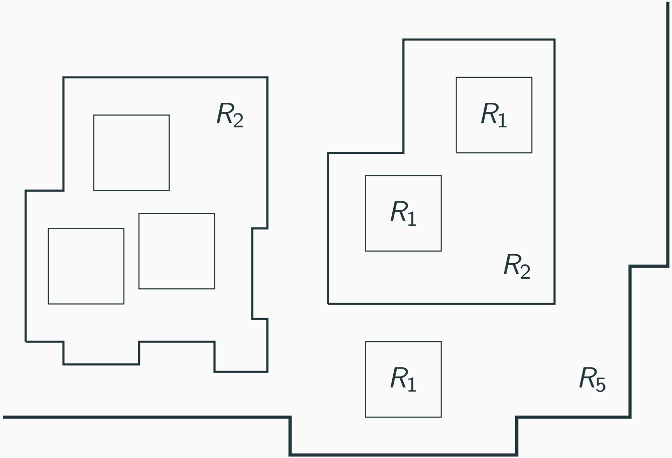

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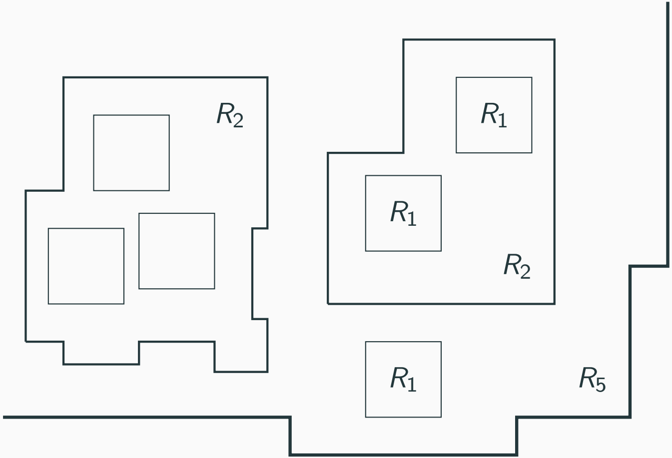

Unlayered Toasts in Borel Dynamics

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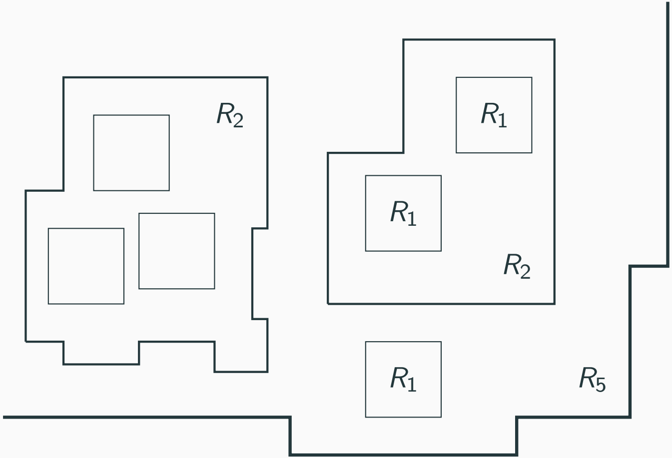

Properties of Regions

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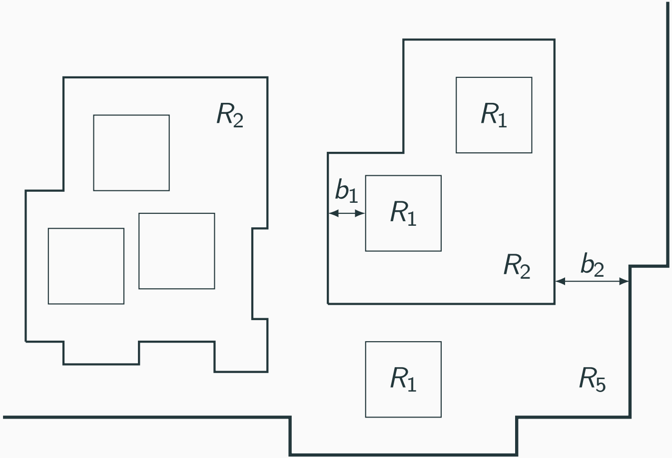
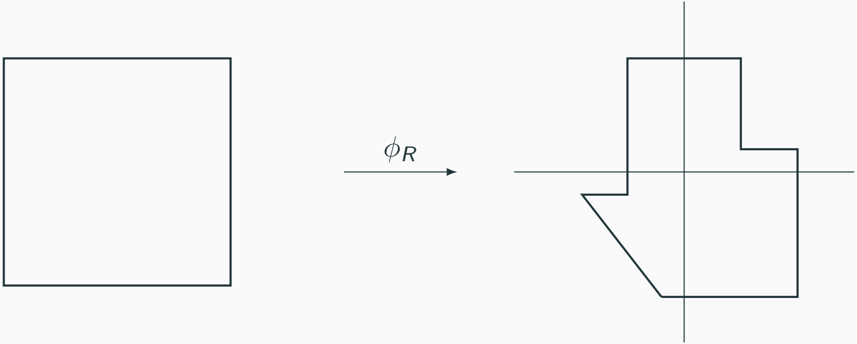


Construction of an Orbit Equivalent Flow

Let R be a region of an unlayered toast. A Borel injection $\phi_R : R \to \mathbb{R}^d$ defines a **partial** action on R

$$x \oplus \vec{r} = \phi_R^{-1}(\phi_R(x) + \vec{r}).$$



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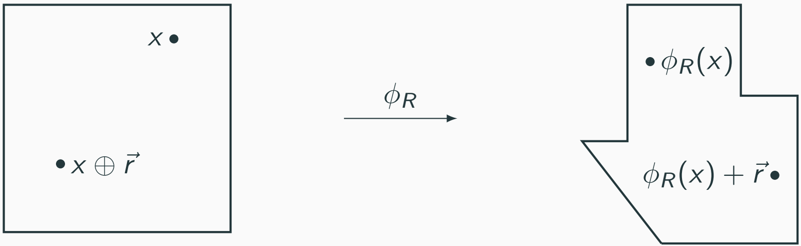

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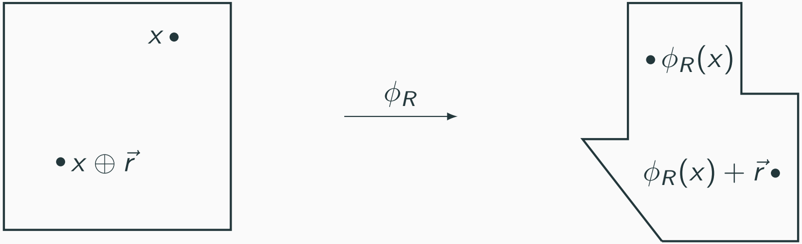

The action is **partial** in the sense that $x \oplus (\vec{r} + \vec{s}) = (x \oplus \vec{r}) \oplus \vec{s}$ whenever both sides are defined.

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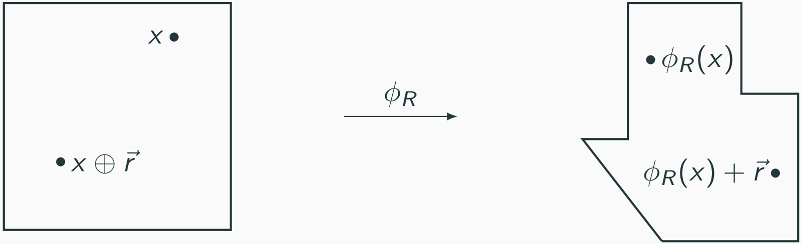

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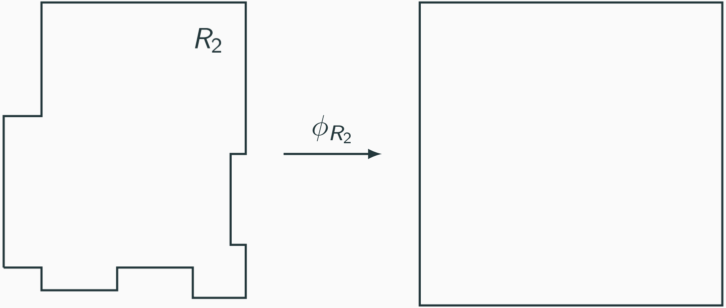
NB: Shifted map $x \mapsto \phi_R(x) + \vec{s}$ defines **the same** partial action for any $\vec{s} \in \mathbb{R}^d$.

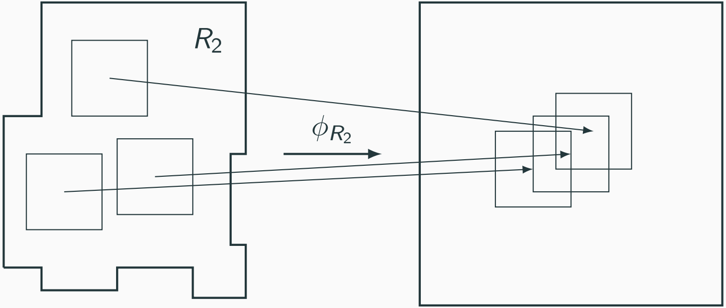
Construction of an Orbit Equivalent Flow

When constructing ϕ_{R_2}, we take into account partial actions given by ϕ_{R_1} for the regions $R_1 \subseteq R_2$.



Construction of an Orbit Equivalent Flow

When constructing ϕ_{R_2}, we take into account partial actions given by ϕ_{R_1} for the regions $R_1 \subseteq R_2$. However, ϕ_{R_2} is **not** an extension of ϕ_{R_1}.



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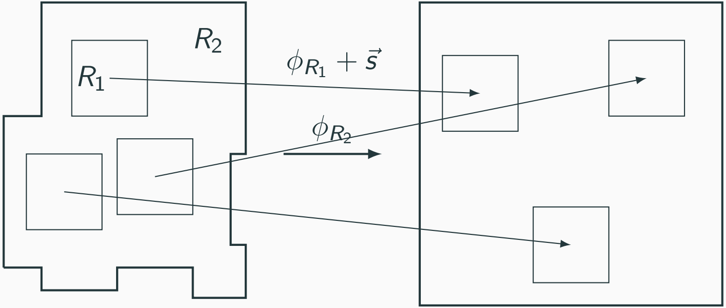
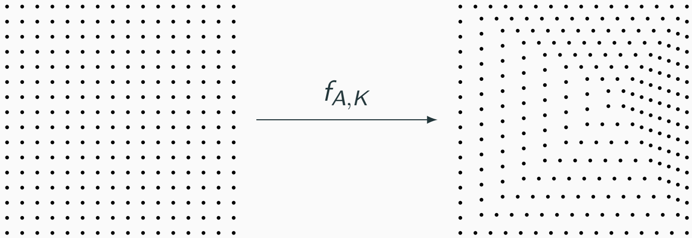


Proof of Borel Version of Katok's Theorem

Lipschitz Maps

Let $\vec{v} \in \mathbb{R}^d$ be of norm $||\vec{v}|| \leq 1$, $K > 1$, and $A \subset \mathbb{R}^d$ be a closed bounded set. Define $f_{A,K} : A \to A$ by

$$f_{A,K}(\vec{r}) = \vec{r} + \frac{d(\vec{r}, \partial A)}{K}\vec{v}.$$



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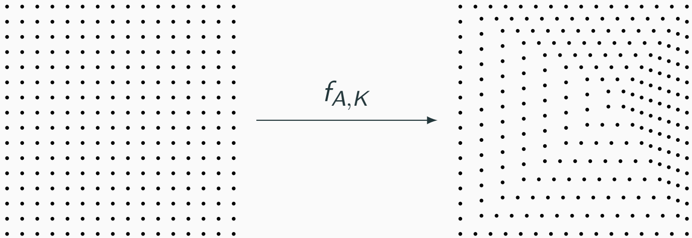
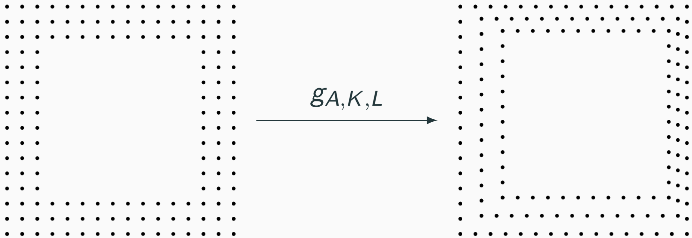


$f_{A,K}$ is $(1 - K^{-1}, 1 + K^{-1})$-bi-Lipschitz and $f_{A,K}(A) = A$.

Lipschitz Maps

Let $A^L = \{\vec{r} \in A : d(\vec{r}, \partial A) \geq L\}$ and observe that $f_{A,K}|_{\partial A^L} = \vec{r} + L/K \cdot \vec{v}$. Define

$$g_{A,K,L}(\vec{r}) = \begin{cases} f_{A,K}(\vec{r}) & \text{if } \vec{r} \in A \setminus A^L; \\ \vec{r} + L/K \cdot \vec{v} & \text{if } \vec{r} \in A^L; \end{cases}$$



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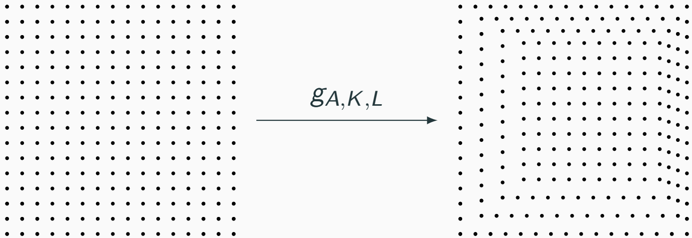

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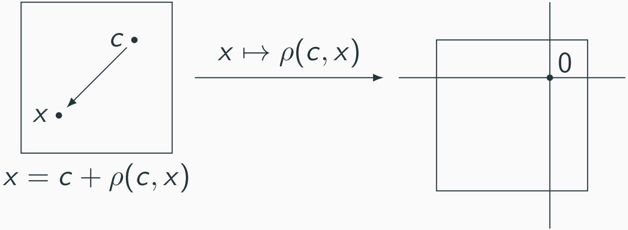
Construction of an Orbit Equivalent Flow

Pick a sequence of unlayered toasts whose boundaries are K-separated for some sufficiently large $K = K(\epsilon)$. We construct a grid that is bi-Lipschitz equivalent to the standard $\mathbb{Z}^d$ grid.

The first step of the construction consists of "identity" maps.



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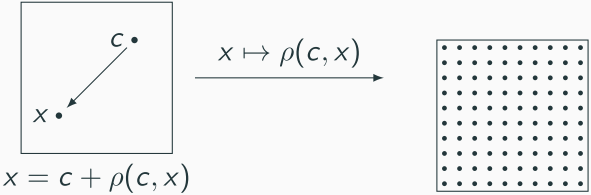

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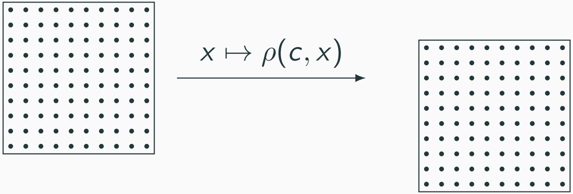
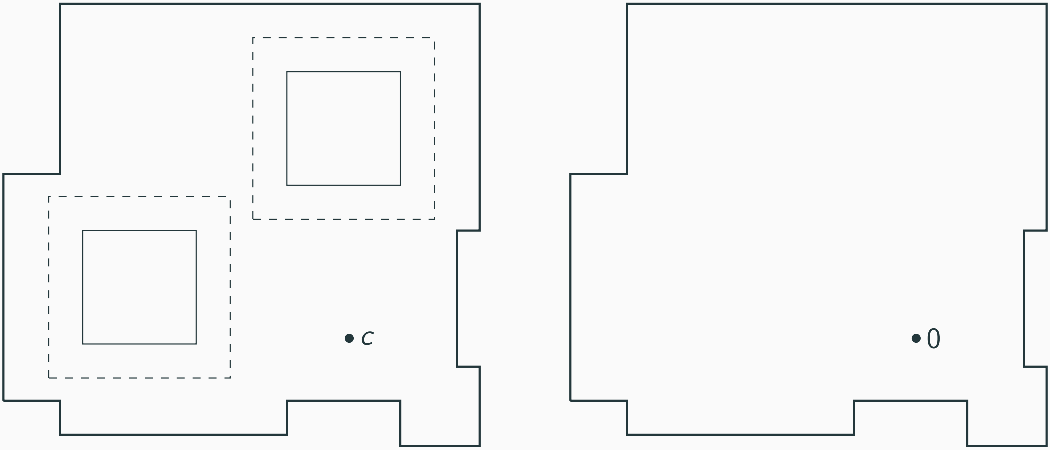


Step of Induction



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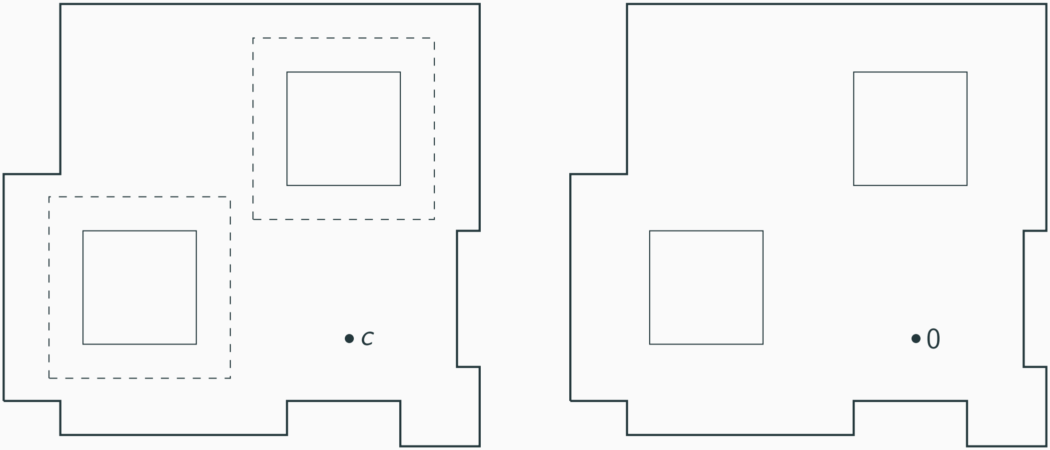

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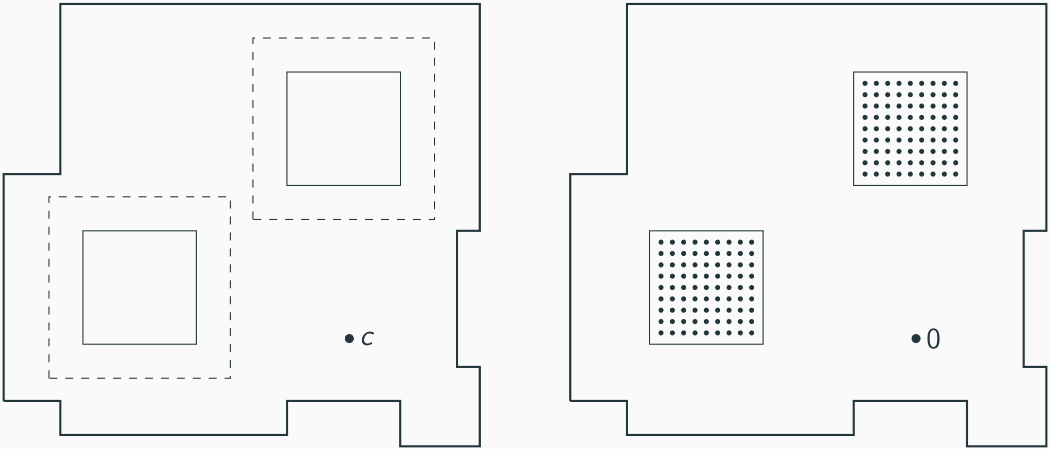

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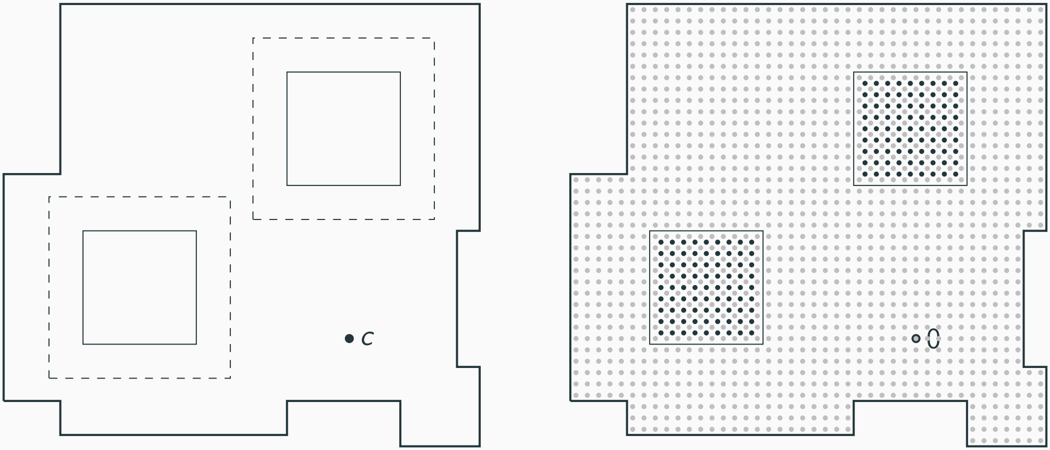

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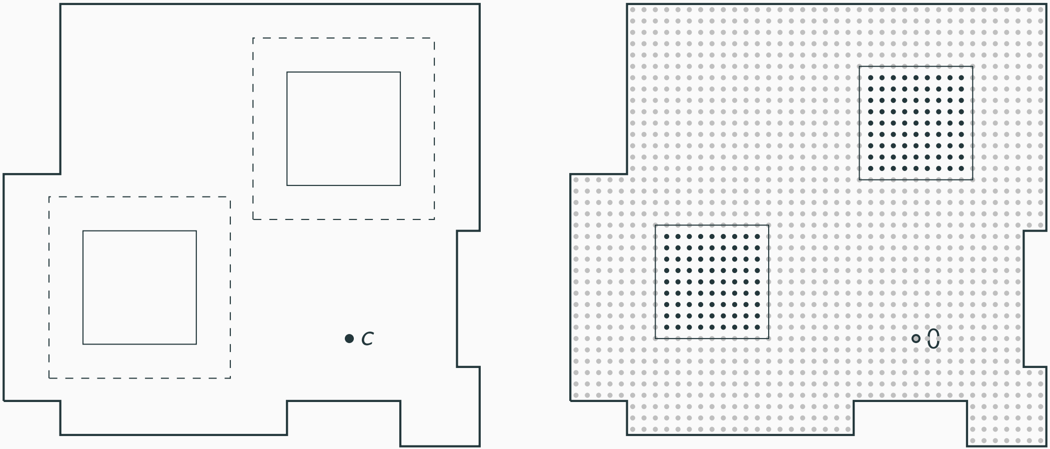

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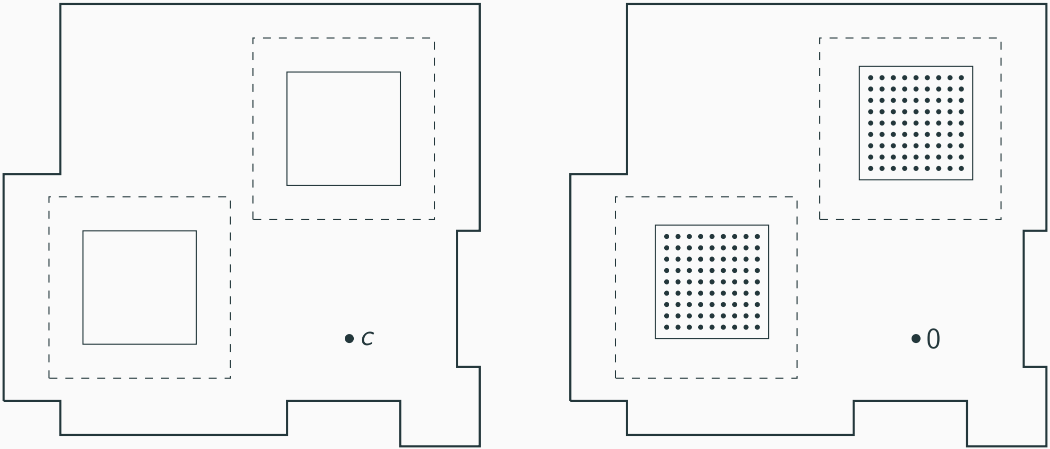

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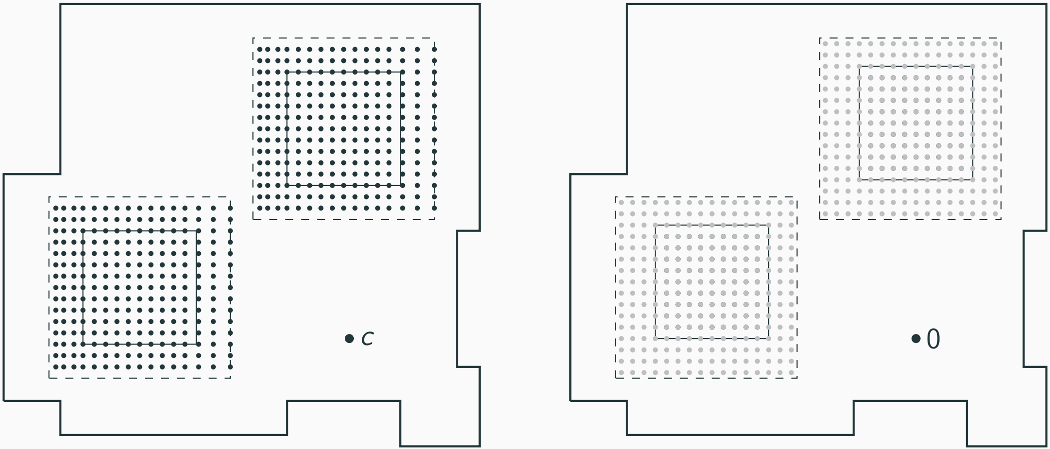

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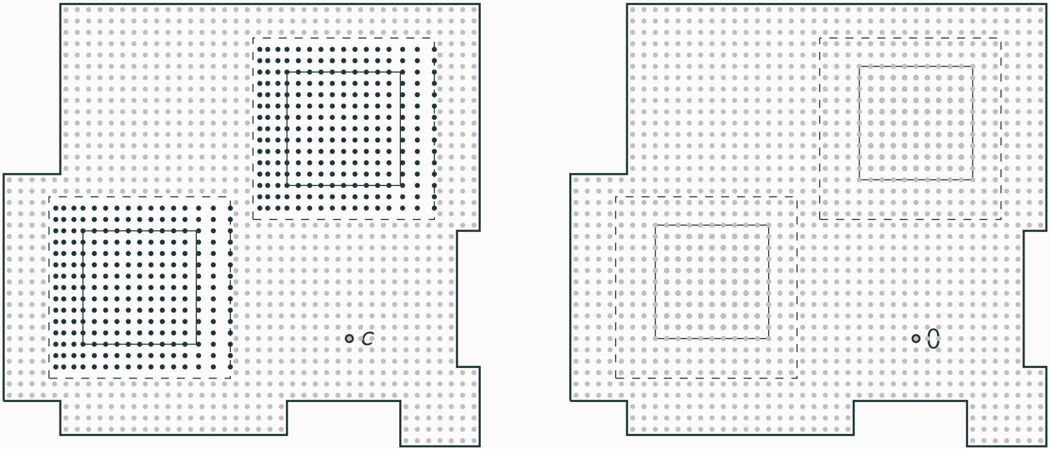

Proof of Borel Version of Rudolph's Theorem

Equivalence to Product Flows

The plan is to reduce the multidimensional case to $d = 1$ by proving the following.

Theorem (S.)

Every free $\mathbb{R}^d$ flow on Ω is **smoothly orbit equivalent** *to a flow $\mathbb{R} \times \mathbb{R}^{d-1} \curvearrowright L \times \mathbb{R}^{d-1}$, where $\mathbb{R} \curvearrowright L$ is one-dimensional, and $\mathbb{R}^{d-1} \curvearrowright \mathbb{R}^{d-1}$ acts by translation.*

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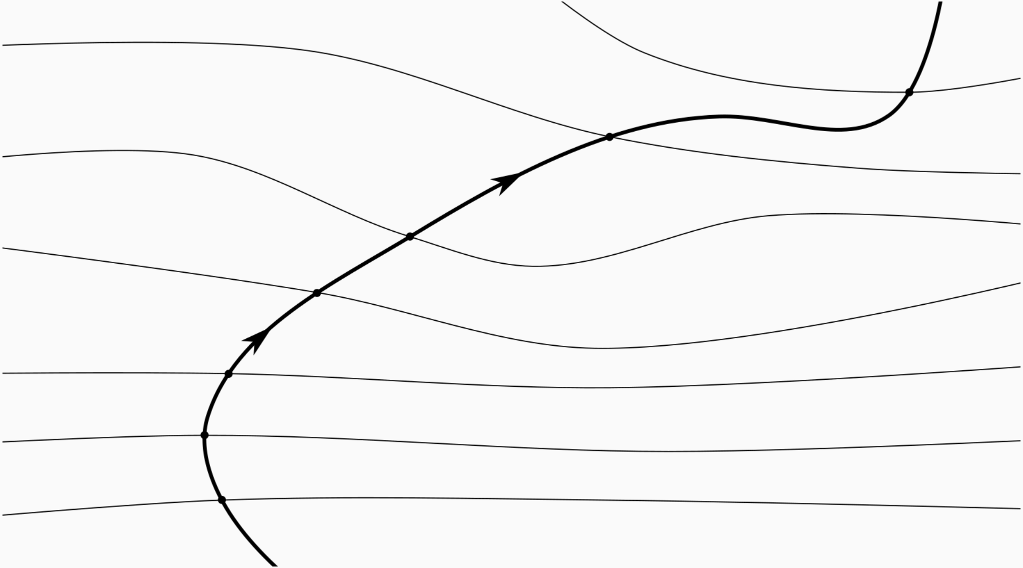
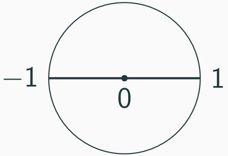
Note that $L \times \vec{0}$ picks a line out of every orbit upon which the $\mathbb{R}$ flow acts.



Equivalence to Product Flows: Base Step

Regions R_1 can be chosen to be diffeomorphic to a unit disk $B_1 \subset \mathbb{R}^d$, so we may pick such a diffeomorphism and pull the line segment $[-1, 1] \times \vec{0}$ into the region R_1 to be part of the line.



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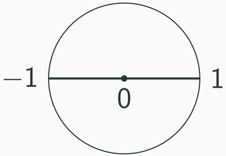

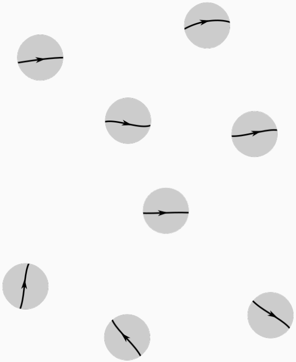

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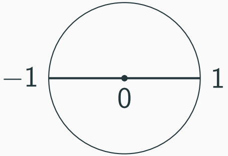
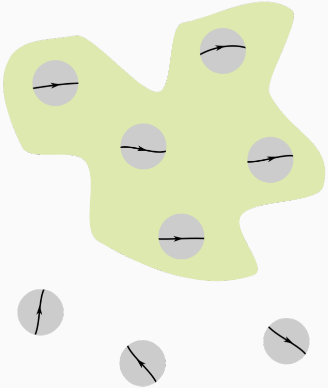




Extending Diffeomorphisms Between Disks

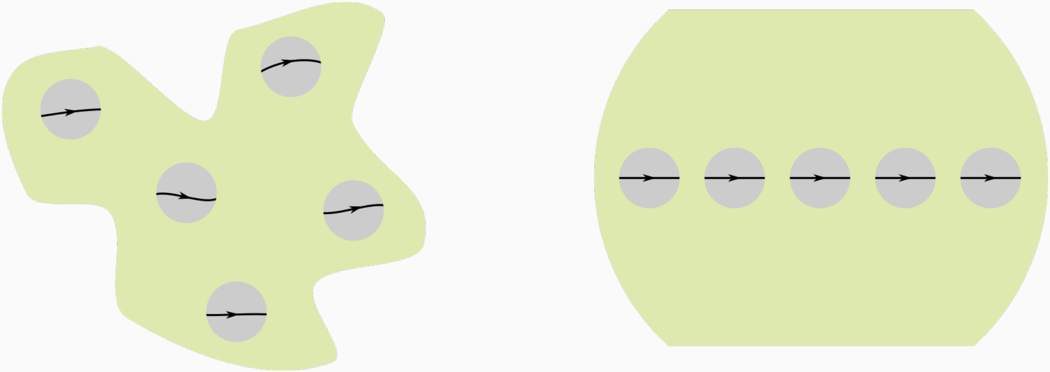
The following basic fact from differential topology is used in the construction.

Theorem

Let F, F' and $D_i \subset F$, $D_i' \subset F'$, $1 \leq i \leq n$, be smooth disks. Suppose that D_i are pairwise disjoint, and so are D_i'. Any family $\phi_i : D_i \to D_i'$ of orientation preserving smooth diffeomorphisms admits a common extension to a diffeomorphism $\psi : F \to F'$.

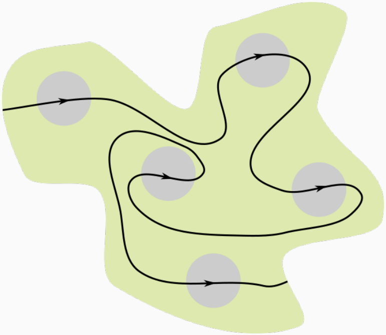
Equivalence of Product Flows: Extending Partial Actions

This lets us extend partial actions on R_1 regions to a R_2 region.



Equivalence to Product Flows: Base Step

The result of such an extension is a partial action on R_2, which extends partial actions on R_1.



Smooth Equivalence of Flows

Let $\mathbb{R}^d \curvearrowright \Omega_1$ and $\mathbb{R}^d \curvearrowright \Omega_2$ be free non tame Borel flows. By the argument above, each of them is smoothly equivalent to a product flow on $L_i \times \mathbb{R}^{d-1}$. The "first coordinate flows" are time change equivalent by the Miller–Rosendal theorem. If $\xi : L_1 \to L_2$ is such a time change equivalence, then

$$L_1 \times \mathbb{R}^{d-1} \ni (y, \vec{r}) \mapsto \big(\xi(y), \vec{r}\big) \in L_2 \times \mathbb{R}^{d-1}.$$

is a smooth equivalence of the multidimensional flows.

Summary of the Results on Smooth Equivalence

This concludes our sketch of the argument.

Dimension	$d = 1$	$d \geq 2$
Ergodic Theory	Many	One
Borel Dynamics	One	**One**

Thank you!