Filter Flows

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Generalizing edges

- An edge $e = (u, v)$ in a graph G can be associated to a pair of filters: namely the principal ultrafilters $\mathcal{U}_u$ and $\mathcal{U}_v$ determined by the vertices u, v on each side of the edge.
- e is on the boundary of a set of vertices $X \subseteq V(G)$ when one of u, v is in X and the other is not.
- ...or, in the language of these filters: when X is measure 1 with respect to one of $\mathcal{U}_u$, $\mathcal{U}_v$ and null with respect to the other.

Generalizing graphs

- Thinking of a usual graph as a network of edges $e = (u, v)$, we'll define a *filter graph* to be a network of filter pairs $(\mathcal{F}, \mathcal{G})$.
- We can make sense of what it means for such an "edge" $(\mathcal{F}, \mathcal{G})$ to be on the boundary of a set of vertices.
- Our goal is to describe how filter graphs resemble graphs in at least one way: the max-flow/min-cut theorem holds for filter graphs.

Outline

1. Flows in graphs
2. Submodular functions
3. Flows in hypergraphs
4. Flows in filter graphs

Flows in Graphs

A prototype flow: König's lemma

Theorem (edge-path version of König's lemma)

Suppose that G is a locally finite graph and x is vertex in G belonging to an infinite connected component of G. There is a neighbor y of x such that if e is the edge connecting x and y, then y belongs to an infinite connected component of $G - e$.

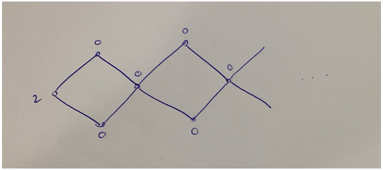
Flows

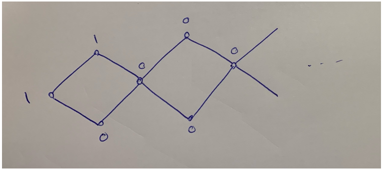
The usual statement of König's lemma follows: every x belonging to an infinite component of G is the initial vertex in some infinite edge-path through G.

Our goal is to generalize König's lemma to produce systems of disjoint paths, in graphs and generalizations of graphs.

Mass assignments

- Suppose G is a locally finite graph.
- A *mass assignment* is a function $u : V(G) \to \mathbb{N}$





Two mass assignments of total mass 2.

Mass assignments are measures

- We may view a mass assignment u as a function on sets of vertices by defining $u(X) = \sum_{x \in X} u(x)$ for $X \subseteq V(G)$.
- So extended, u is a measure on $V(G)$.
- The finite additivity of u is equivalent to another condition called *modularity*:

$$u(X \cup Y) + u(X \cap Y) = u(X) + u(Y)$$

for all $X, Y \subseteq V(G)$.

Feasible mass assignments

Question: Which mass assignments allow us to send the units of mass along pairwise edge-disjoint paths?

Boundaries

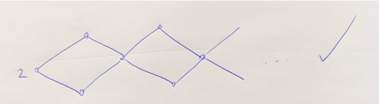
- Given a set of vertices $X \subseteq V(G)$, the edge boundary function is defined by:

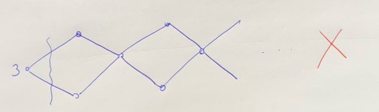
$$\begin{aligned} \partial_G(X) &= \#\text{ of edges on the boundary of } X \\ &= |\{e \in E(G) : \text{exactly one end of } e \text{ is in } X\}| \end{aligned}$$

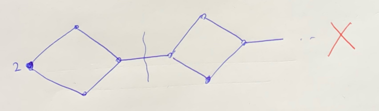
- If u is a mass assignment, and for some X we have $u(X) > \partial_G(X)$, we can't possibly send the units of mass in X along edge-disjoint paths out of X.
- The max-flow/min-cut theorem says this is the only restriction to finding such a system of paths.

Feasible mass assignments

- A mass assignment is called *feasible* if for every $X \subseteq V(G)$ we have $u(X) \leq \partial_G(X)$.







Push and burn

Theorem (max-flow/min-cut)

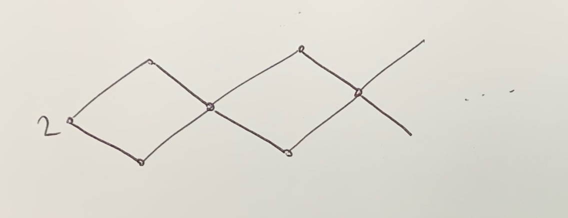
Suppose that G is locally finite graph and u is a feasible mass assignment for G.

Fix $x \in V(G)$ with $u(x) \geq 1$. There is a neighbor y of x to which we can push a unit of mass from x and burn the connecting edge.

That is, if e is the edge connecting x and y and we define a new mass assignment u' by:

$$\begin{aligned} u'(x) &= u(x) - 1 \\ u'(y) &= u(y) + 1 \\ u'(v) &= u(v) \text{ for all } v \neq x, y, \end{aligned}$$

Then u' is a feasible mass assignment for $G' = G - e$.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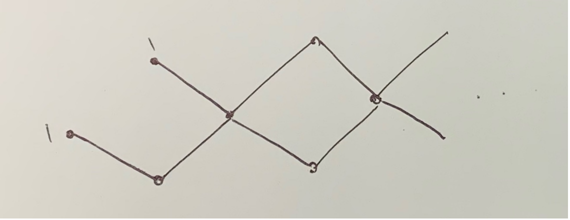
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Submodular Functions

The key for flows: submodularity of ∂_G

The proof of max-flow/min-cut depends crucially on the *submodularity* of the boundary function ∂_G.

Let's forget about graphs for the moment and approach submodularity abstractly.

We'll see that certain simple submodular functions resemble edges in a graph.

Submodularity

Suppose that V is a set, and let 2^V denote its powerset.

Definition

A function $f : 2^V \to \mathbb{R}$ is called *submodular* if for all $X, Y \subseteq V$ we have

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y).$$

- All submodular functions we consider will be non-negative integer valued, i.e. we'll have $f : 2^V \to \mathbb{N}$.
- We will often have $f(\emptyset) = 0$.

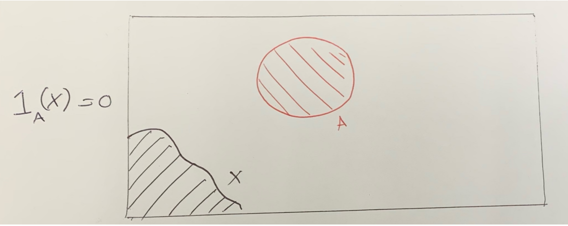
Intersect to increase

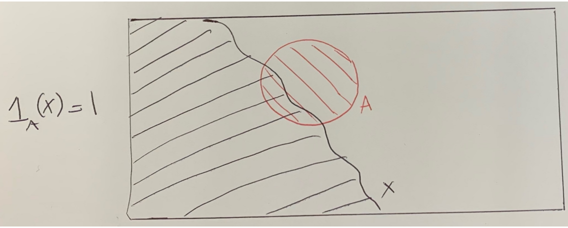
Example

Fix $A \subseteq V$. Define a function 1_A on 2^V by

$$\begin{aligned} 1_A(X) &= 0 \quad \text{if } A \cap X = \emptyset, \\ 1_A(X) &= 1 \quad \text{if } A \cap X \neq \emptyset. \end{aligned}$$

Then 1_A is submodular.


$\mathbb{1}_A(x) = 0$
A
x


$\mathbb{1}_A(x) = 1$
A
x

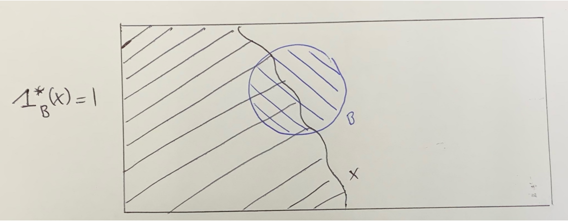
Contain to decrease

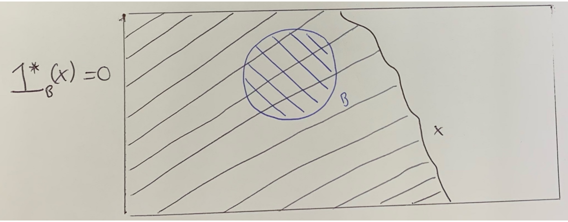
Example

Fix $B \subseteq V$. Define a function 1_B^* on 2^V by

$$\begin{aligned} 1_B^*(X) &= 1 \quad \text{if } B \not\subseteq X, \\ 1_B^*(X) &= 0 \quad \text{if } B \subseteq X. \end{aligned}$$

Then 1_B^* is submodular.


$\mathbb{1}^*_B(x) = 1$
B
x


$\mathbb{1}^*_B(x) = 0$
B
x

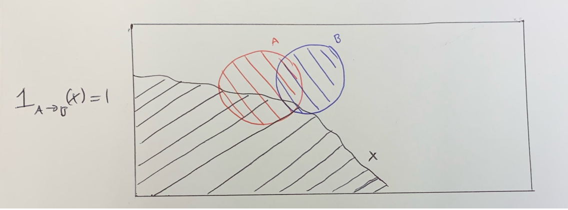
Up then down

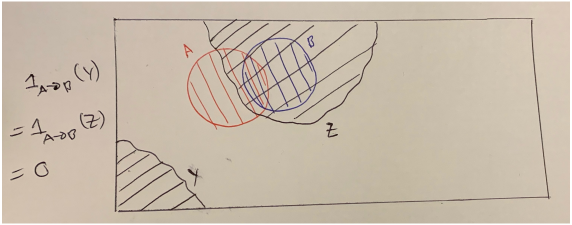
Example

Fix $A, B \subseteq V$, not both equal to the same singleton. Define a function $1_{A \to B}$ on 2^V by

$$\begin{aligned} 1_{A\to B}(X) &= 0 \quad \text{if } A \cap X = \emptyset \text{ or } B \subseteq X, \\ 1_{A\to B}(X) &= 1 \quad \text{if } A \cap X \neq \emptyset \text{ and } B \not\subseteq X. \end{aligned}$$

Then $1_{A \to B}$ is submodular.


$1_{A\to B}(x) = 1$
A
B
x


$1_{A\to B}(Y)$
$= 1_{A\to B}(Z)$
$= 0$
A
B
Z
Y

2-valued submodular functions

It turns out these are the only non-constant examples of 2-valued submodular functions. . . on finite domains.

Theorem

Suppose that V is a finite set and $f : 2^V \to \{0, 1\}$ is a submodular function. Then exactly one of the following holds:

(i) $f(X) = 0$ *for all* $X \subseteq V$,

(ii) $f(X) = 1$ *for all* $X \subseteq V$,

(iii) $f = 1_A$ *for some nonempty* $A \subseteq V$,

(iv) $f = 1^*_B$ *for some nonempty* $B \subseteq V$,

(v) $f = 1_{A \to B}$ *for some nonempty* $A, B \subseteq V$ *not both equal to the same singleton.*

More complicated submodular functions

Submodular functions with more than two output values are harder to describe.

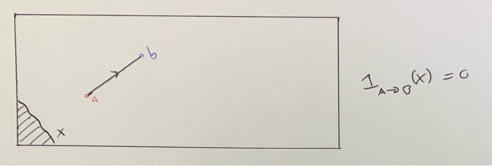
But the intuition from the 2-valued situation generalizes: if f is submodular on a finite domain V, then f increases as the input set X becomes incident to certain subsets of V, and decreases when X finally contains certain subsets.

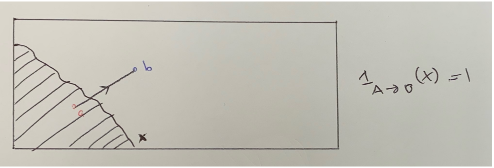
Flows in Hypergraphs

$1_{A \to B}$ as an edge indicator

Let's consider submodular functions of the form $1_{A \to B}$.

- If $A = \{a\}$ and $B = \{b\}$ for some distinct $a, b \in V$, we can think of the pair (a, b) as a directed edge from a to b.
- Then $1_{A \to B}$ indicates whether this edge is on the (outgoing) edge boundary of the input set X.


b
a
x
$\mathbb{1}_{A\to B}(x) = 0$


b
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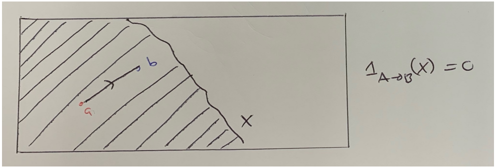
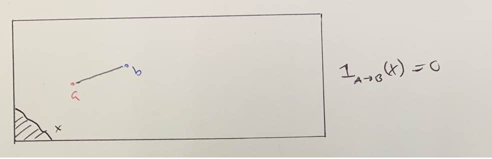
$\mathbb{1}_{A\to B}(x) = 1$


b
a
x
$\mathbb{1}_{A\to B}(x) = 0$

$1_{A \to B}$ as an edge indicator

- If $A = B = \{a, b\}$ for some distinct $a, b \in V$, we can think of the pair $\{a, b\}$ as an undirected edge between a and b.
- Then $1_{A \to B}$ indicates whether this edge is on the edge boundary of the input set X.


a
b
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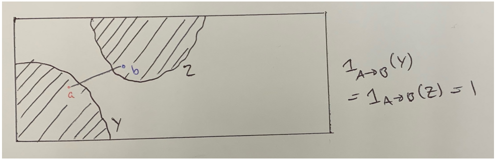
1A→B(x) = 0


a
b
Y
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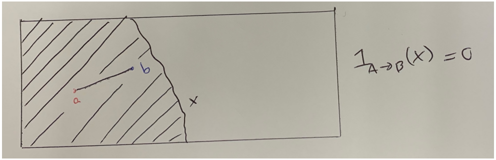
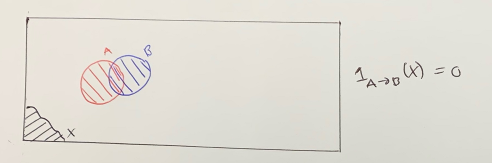
1A→B(Y)
= 1A→B(Z) = 1


a
b
x
1A→B(x) = 0

$1_{A \to B}$ as an edge indicator

- In general we can think of a pair of subsets (A, B) as a directed hyperedge from A to B, and the function $1_{A \to B}$ as indicating whether this hyperedge is on the outgoing boundary of the input set.


A
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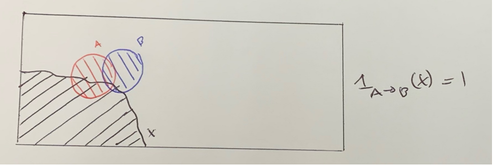
$\mathbb{1}_{A\to B}(x) = 0$


A
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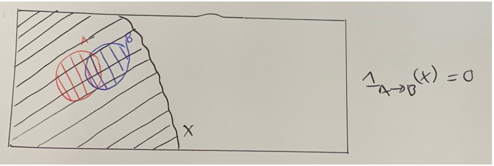
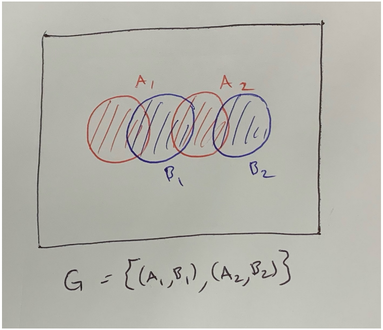
$\mathbb{1}_{A\to B}(x) = 1$


A
B
X
$\mathbb{1}_{A\to B}(x) = 0$

Directed hypergraphs

- We think of a collection of directed hyperedges $G = \{(A_i, B_i)\}$ as a directed hypergraph.
- Then the function $F = \sum_i 1_{A_i \to B_i}$ is the outgoing edge boundary function for G.



(Sources and sinks)

- (A lone subset A indicated by its associated function 1_A can be thought of as a *sink*, and a B indicated by 1^*_B can be thought of as a *source*.)
- (For simplicity, we won't include sources and sinks in our graphs, instead imagining all paths as flowing out to infinity.)

Sums of submodular functions are submodular

Fact: Given a collection of submodular functions f_i and non-negative real numbers a_i, the function $F = \sum_i a_i f_i$ is submodular.

Hence if $G = \{(A_i, B_i)\}$ is a directed hypergraph and $F = \sum_i 1_{A_i \to B_i}$ is its outgoing edge boundary function, then F is submodular.

In particular, the edge boundary function ∂ of an undirected graph G is submodular.

Max-flow/min-cut for hypergraphs

We can generalize the max-flow/min-cut theorem to directed hypergraphs.

First we need to generalize the notion of a feasible mass assignment.

If $F : 2^V \to \mathbb{N}$ is submodular (e.g. the edge boundary function for a directed hypergraph), a mass assignment $u : V \to \mathbb{N}$ is *feasible* if $u(X) \leq F(X)$ for all $X \subseteq V$.

Saturated sets form a lattice

We need only one fact about submodularity in the proof.

If u is a feasible mass assignment for a submodular F, we call a set $X \subseteq V$ *saturated* if $u(X) = F(X)$.

Fact. The collection of saturated sets $\{X \subseteq V : u(X) = F(X)\}$ is closed under $\cap$ and $\cup$.

This will allow us to make a move akin to "the union of a finite collection of finite sets is finite" that we make in the proof of König's lemma.

Max-flow/min-cut for hypergraphs

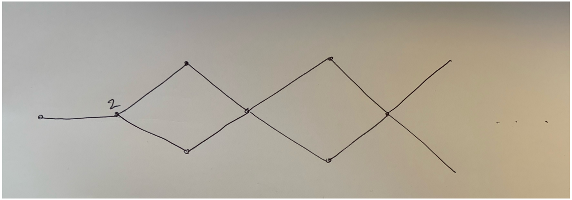
Theorem

Suppose that $G = \{(A_i, B_i)\}$ *is a locally finite directed hypergraph with vertex set* V *and edge boundary function* $F = \sum_i 1_{A_i \to B_i}$. *Suppose* $u : V \to \mathbb{N}$ *is a feasible mass assignment for* F.

Fix $x \in V$ *with* $u(x) \geq 1$. *Then for some edge* $(A, B) \in G$ *with* $x \in A$, *there is a vertex* $y \in B$ *such that if we define a new mass assignment* u' *by:*

$$\begin{aligned} u'(x) &= u(x) - 1 \\ u'(y) &= u(y) + 1 \\ u'(v) &= u(v) \text{ for all } v \neq x, y, \end{aligned}$$

Then u' *is feasible for* $F' = F - 1_{A \to B}$.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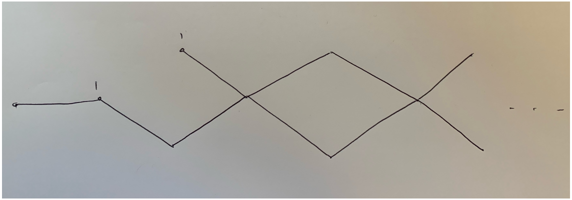
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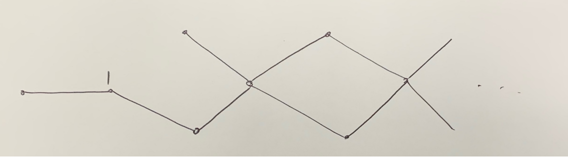
Proof

Step 1: "Pick up one unit mass and burn the edge"

- List the edges $(A_1, B_1), \ldots, (A_n, B_n)$ for which $x \in A_i$.
- Let u^* denote the mass assignment that removes a unit of mass from x.
- **Claim**: There is $i \leq n$ s.t. u^* is feasible for $F' = F - 1_{A_i \to B_i}$.



Proof

- If not, can check: for every $i \leq n$ there is a saturated X_i that intersects A_i but doesn't contain x.
- Let $X = \bigcup_{i \leq n} X_i$. Then X is saturated, i.e. $F(X) = u(X)$.
- Since X intersects every A_i containing x, we have $F(X \cup \{x\}) = F(X)$.
- Hence $F(X \cup \{x\}) = u(X)$.
- But $u(X \cup \{x\}) > u(X)$, since u assigns at least one unit of mass to x.
- So then $u(X \cup \{x\}) > F(X \cup \{x\})$, contradicting feasibility.

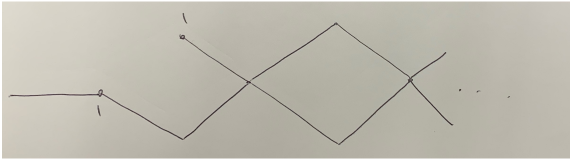
Proof

Short version: If we can't pick up a unit mass at x and burn one of the edges incident to x, we find a saturated set X that stakes a claim for all these edges along which the mass on x might escape, but that doesn't contain x — impossible.

Proof

Step 2: "Put the mass back down on the other side"

- Reordering if need be, suppose $(A_1, B_1), \dots, (A_m, B_m)$ are those edges above which allow a "pick up and burn" move.
- **Claim**: there is $j \leq m$ and $y \in B_j$ s.t. if we let u' denote the mass assignment which reassigns the deleted mass to y, then u' is feasible for F'.



Proof

- If not, can check: for every $j \leq m$ and every $y \in B_j$ there is a $Y_{j,y}$ containing y that is F'-saturated wrt u^*.
- Then $Y_j = \bigcup_{y \in B_j} Y_{j,y}$ is also F'-saturated wrt u^*.
- One can check: because Y_j contains B_j, it follows Y_j is saturated in the original sense: that is, F-saturated wrt u.
- Hence so is $Y = \bigcup_{j \leq m} Y_j$. We also have that Y cannot contain x.

Proof

- Hence $X \cup Y$ is also saturated, that is $u(X \cup Y) = F(X \cup Y)$, and moreover doesn't contain x.
- X intersects some of the A_i to which x belongs. For the remaining A_j, Y completely contains the corresponding B_j.
- Hence $F(X \cup Y \cup \{x\}) = F(X \cup Y)$.
- But $u(X \cup Y \cup \{x\}) > u(X \cup Y)$ since u assigns at least one unit of mass to x.
- So $u(X \cup Y \cup \{x\}) > F(X \cup Y \cup \{x\})$, contradicting feasibility.

Proof

Short version: If we can't put down the unit mass we picked up at x on some neighbor y, we find a new saturated set Y which contains all of these potential landing spots and doesn't contain x. Then $X \cup Y$ either stakes a claim for the edges along which the mass at x might escape, or contains those edges internally, and yet doesn't contain x — impossible.

Flows in Filter Graphs

Generalizing max-flow/min-cut

So far:

- Non-constant 2-valued submodular functions on a *finite* domain must be of the form 1_A, 1_B^* or $1_{A \to B}$.
- Max-flow/min-cut holds for networks of these functions (directed hypergraphs).

What about 2-valued submodular functions on an *infinite* domain?

Filters

Suppose V is a set.

Recall: a *filter* on V is a collection of subsets $\mathcal{F} \subseteq 2^V$ such that

1. $V \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$.
2. $X \in \mathcal{F}$ and $Y \supseteq X$ implies $Y \in \mathcal{F}$.
3. $X, Y \in \mathcal{F}$ implies $X \cap Y \in \mathcal{F}$.

For $X \subseteq V$, we say X is $\mathcal{F}$-*null* if $\overline{X} \in \mathcal{F}$; X is $\mathcal{F}$-*positive* if $\overline{X} \notin \mathcal{F}$.

A filter $\mathcal{F}$ is an *ultrafilter* if for every $X \subseteq V$ either $X \in \mathcal{F}$ or $\overline{X} \in \mathcal{F}$.

Principal filters and ultrafilters

Example

Fix $A \subseteq V$.

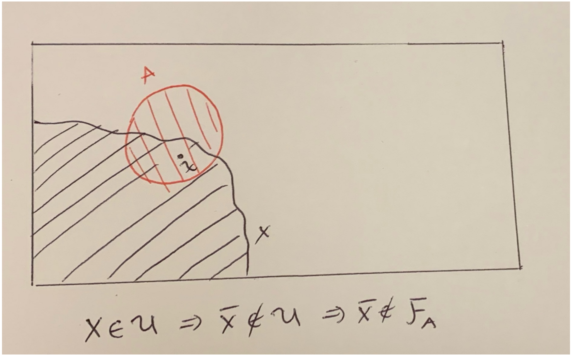
- The collection $\mathcal{F} = \{X \subseteq V : A \subseteq X\}$ is a filter on V, called a principal filter.
- The $\mathcal{F}$-null sets are those X such that $X \cap A = \emptyset$, the $\mathcal{F}$-positive sets are those X such that $X \cap A \neq \emptyset$.
- If $A = \{a\}$ is a singleton, $\mathcal{F}$ is an ultrafilter.
- If V is finite, the principal filters are the only filters on V.

Extending a filter to an ultrafilter :: placing a unit of mass

A good picture to keep in mind:

- Suppose A is a subset of V and $\mathcal{F}_A$ is the associated principal filter.
- If we extend $\mathcal{F}_A$ to an ultrafilter $\mathcal{U}$, then $\mathcal{U}$ is a principal ultrafilter determined by some $x \in A$.
- If A is one of the sets in a directed hyperedge (A, B), extending $\mathcal{F}_A$ to $\mathcal{U}$ is tantamount to placing a unit of mass on the outgoing side of this edge.

Extending a filter to an ultrafilter :: placing a unit of mass



Become positive to increase

Suppose V is infinite. What are the 2-valued submodular functions on V?

Example

Fix a filter $\mathcal{F}$ on V. Define a function $1_{\mathcal{F}}$ on 2^V by

$$\begin{array}{rcl} 1_{\mathcal{F}}(X) & = & 0 \quad \text{if } \overline{X} \in \mathcal{F}, \\ 1_{\mathcal{F}}(X) & = & 1 \quad \text{if } \overline{X} \notin \mathcal{F}. \end{array}$$

Then $1_{\mathcal{F}}$ is submodular.

Functions of this form generalize functions of the form 1_A.

Become co-null to decrease

Example

Fix a filter $\mathcal{G}$ on V. Define a function $1^*_{\mathcal{G}}$ on 2^V by

$$\begin{aligned} 1^*_{\mathcal{G}}(X) &= 1 \quad \text{if } X \notin \mathcal{G}, \\ 1^*_{\mathcal{G}}(X) &= 0 \quad \text{if } X \in \mathcal{G}. \end{aligned}$$

Then $1^*_{\mathcal{G}}$ is submodular.

Up then down

Example

Fix filters $\mathcal{F}, \mathcal{G}$ on V, not both equal to the same ultrafilter. Define a function $1_{\mathcal{F}\to\mathcal{G}}$ by

$$\begin{array}{rcll} 1_{\mathcal{F}\to\mathcal{G}}(X) & = & 0 & \text{if } \overline{X} \in \mathcal{F} \text{ or } X \in \mathcal{G}, \\ 1_{\mathcal{F}\to\mathcal{G}}(X) & = & 1 & \text{if } \overline{X} \notin \mathcal{F} \text{ and } X \notin \mathcal{G}. \end{array}$$

Then $1_{\mathcal{F}\to\mathcal{G}}$ is submodular.

2-valued submodular functions

These are the only non-constant examples of 2-valued submodular functions—full stop!

Theorem

Suppose that V is a set and $f : 2^V \to \{0, 1\}$ is a submodular function. Then exactly one of the following holds:

(i) $f(X) = 0$ *for all* $X \subseteq V$*,*

(ii) $f(X) = 1$ *for all* $X \subseteq V$*,*

(iii) $f = 1_{\mathcal{F}}$ *for some filter* $\mathcal{F}$ *on* V*,*

(iv) $f = 1^*_{\mathcal{G}}$ *for some filter* $\mathcal{G}$ *on* V*,*

(v) $f = 1_{\mathcal{F} \to \mathcal{G}}$ *for some filters* $\mathcal{F}, \mathcal{G}$ *on* V*, not both equal to the same ultrafilter.*

Filter graphs

- We can think of a pair of filters $(\mathcal{F}, \mathcal{G})$ on V as a generalization of a directed edge in a hypergraph.
- For $X \subseteq V$, we think of the value $1_{\mathcal{F} \to \mathcal{G}}(X)$ as indicating whether this edge is on the outgoing boundary of X.
- Let's call a collection of these edges $G = \{(\mathcal{F}_i, \mathcal{G}_i)\}$ a *filter graph* on V.
- Then the function $F = \sum_i 1_{\mathcal{F}_i \to \mathcal{G}_i}$ is the edge boundary function for G (and is submodular too).

An example

Example

- Suppose A, B, C are infinite pairwise disjoint subsets of V.
- Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be, respectively, the cofinite filters on A, B, C, closed upward so as to be filters on V.
- Then $G = \{(\mathcal{F}, \mathcal{G}), (\mathcal{G}, \mathcal{H})\}$ is a filter graph with two edges, pointing sets that are infinite on A to sets infinite on B, and these to sets infinite on C in turn.

Mass assignments are measures

- A *mass assignment* is a modular function $u : 2^V \to \mathbb{R}$.
- All of our mass assignments will be finitely additive measures (i.e. increasing and non-negative).
- Given an ultrafilter $\mathcal{U}$ on V, we define the usual associated measure $u : 2^V \to \{0, 1\}$ by $u(X) = 1$ iff $X \in \mathcal{U}$.
- Actually, our mass assignments will be of the form $u = \sum u_i$, where u_i are the measures associated to a collection of ultrafilters $\mathcal{U}_i$.

A "point-mass" ready to cross over an "edge"

Example

- Consider again $G = \{(\mathcal{F}, \mathcal{G}), (\mathcal{G}, \mathcal{H})\}$ be our two-edged filter graph from before, associated the to infinite sets A, B, C.
- if $\mathcal{U}$ is an ultrafilter extending $\mathcal{F}$, we might think of $\mathcal{U}$ as being a "point mass" on the A-side of the edge $(\mathcal{F}, \mathcal{G})$.
- We have $u(X) = 1$ implies $1_{\mathcal{F}}(X) = 1$, which in turn implies $1_{\mathcal{F} \to \mathcal{G}}(X) = 1 \ldots$ as long as X does not belong to $\mathcal{G}$.
- Intuitively, this says that any X containing the mass from $\mathcal{U}$ has the edge $(\mathcal{F}, \mathcal{G})$ on its boundary, unless this edge is internal to X.

Max-flow/min-cut for filter graphs

- The definition of *feasible* is the same as before: a mass assignment u is feasible for a submodular F if $u(X) \leq F(X)$ for all $X \subseteq V$.
- If $G = \{(\mathcal{F}_i, \mathcal{G}_i)\}$ is a finite filter graph, and $u = \sum_i u_i$ is a sum of ultrafilter measures (associated to some ultrafilters $\mathcal{U}_i$) which is feasible for the boundary function $F = \sum_i 1_{\mathcal{F}_i \to \mathcal{G}_i}$, then one can check that every $\mathcal{U}_i$ extends some $\mathcal{F}_i$.
- That is, all "point-masses" sit on the outgoing side of at least one "edge."

Max-flow/min-cut for filter graphs

Theorem

Suppose that $G = \{(\mathcal{F}_j, \mathcal{G}_j)\}$ is a finite filter graph with vertex set V and boundary function $F = \sum_i 1_{\mathcal{F}_j \to \mathcal{G}_j}$. Suppose $u = \sum_i u_i$ is a sum of ultrafilter measures (associated to ultrafilters $\mathcal{U}_i$) which is feasible for F.

Fix i, and consider the ultrafilter $\mathcal{U}_i$ and associated measure u_i. For some edge $(\mathcal{F}, \mathcal{G}) \in G$ where $\mathcal{U}_i$ extends $\mathcal{F}$, there is an ultrafilter $\mathcal{U}_i'$ extending $\mathcal{G}$ such that if we define a new mass assignment u' by

$$u' = u - u_i + u_i'$$

Then u' is feasible for $F' = F - 1_{\mathcal{F} \to \mathcal{G}}$.

Proof

Step 1: "Pick up the ultrafilter and burn the edge"

- List the pairs $(\mathcal{F}_1, \mathcal{G}_1), \ldots, (\mathcal{F}_n, \mathcal{G}_n)$ for which $\mathcal{U}_i$ extends $\mathcal{F}_j$.
- Let $u^* = u - u_i$.
- **Claim**: There is $j \leq n$ such that u^* is feasible for $F' = F - 1_{\mathcal{F}_j \to \mathcal{G}_j}$.

Proof

- If not, for every $j \leq n$ we must be able to find a saturated set X_j with $u_i(X_j) = 0$ (i.e. $X_j \notin \mathcal{U}_i$) and with $1_{\mathcal{F}_j \to \mathcal{G}_j}(X_j) = 1$ (i.e. X_j is $\mathcal{F}_j$-positive and $\overline{X_j}$ is $\mathcal{G}_j$-positive).
- Then $X = \bigcup_{j \leq n} X_j$ is also saturated and $\mathcal{U}_i$-null.
- We can find $A \subseteq V$ that sees the mass at $\mathcal{U}_i$ (i.e. $A \in \mathcal{U}_i$, i.e. $u_i(A) = 1$) but isn't incident to any of the edges $(\mathcal{F}, \mathcal{G})$ away from this mass (i.e. A is $\mathcal{F}$-null when $\mathcal{U}_i$ does not extend $\mathcal{F}$).
  - (using: finite intersection of co-null sets is co-null; if an ultrafilter $\mathcal{U}$ does not extend a filter $\mathcal{F}$, then there is an $\mathcal{F}$-null set in $\mathcal{U}$.)

Proof

- Then since X is already incident to all edges incident to $\mathcal{U}_i$, and A is incident only to these, we have $F(X \cup A) = F(X)$.
- Since X is saturated we have $F(X \cup A) = F(X) = u(X)$.
- On the other hand, since X is $\mathcal{U}_i$-null and A is not we have $u(X \cup A) = u(X) + 1$.
- Combining these lines gives $F(X \cup A) < u(X \cup A)$, contradicting feasibility.

Proof

Step 2: "Put an ultrafilter back down on the other side"

- Reordering if necessary, list the pairs $(\mathcal{F}_1, \mathcal{G}_1), \dots, (\mathcal{F}_m, \mathcal{G}_m)$ above for which we can do a "pick up and burn" move (there's at least one).
- **Claim**: There is $j \leq m$ and some ultrafilter $\mathcal{U}_i'$ extending $\mathcal{G}_j$ with associated measure u_i' such that if we define

$$u' = u^* + u_i' = u - u_i + u_i'$$

Then u' is feasible for $F' = F - 1_{\mathcal{F}_j \to \mathcal{G}_j}$.

Proof

- If not, for every $j \leq m$ and every ultrafilter $\mathcal{V}$ extending $\mathcal{G}_j$ we can find a set $Y_{j,v}$ such that $F'(Y_{j,v}) = u^*(Y_{j,v})$.
- Since the space of ultrafilters on V is compact and the collection of ultrafilters extending $\mathcal{G}_j$ is closed in this space, we can actually find an *finite* collection of these saturated sets $Y_{j,v_1}, \ldots, Y_{j,v_p}$ such that every ultrafilter $\mathcal{V}$ extending $\mathcal{G}_j$ contains one of these sets — and hence their union Y_j, which must also be saturated.
- But the only sets which are in *every* ultrafilter extending $\mathcal{G}_j$ are sets in $\mathcal{G}_j$ itself. Hence Y_j is in $\mathcal{G}_j$.
- As before, it follows that not only do we have $F'(Y_j) = u^*(Y_j)$ but actually $F(Y_j) = u^*(Y_j)$.

Proof

- Letting $Y = \bigcup_{j \leq m} Y_j$ we have $F(Y) = u^*(Y)$.
- If Y isn't $\mathcal{U}_i$-null, we immediately contradict feasibility since in this case $u(Y) = u^*(Y) + 1$.
- If Y is $\mathcal{U}_i$-null, then in fact $F(Y) = u(Y)$. We can find as before a set A in $\mathcal{U}_i$ that is $\mathcal{F}_k$ null for all $\mathcal{F}_k$ in edges $(\mathcal{F}_k, \mathcal{G}_k)$ not incident to $\mathcal{U}_i$.
- Since X from part 1 of the proof is saturated we have $X \cup Y$ is too, i.e. $F(X \cup Y) = u(X \cup Y)$.
- Our A is chosen so that $F(X \cup Y \cup A) = F(X \cup Y)$.
- But A is also chosen so that $u_i(A) = 1$.
- Combining yields,
  $F(X \cup Y \cup A) = u(X \cup Y) < u(X \cup Y) + 1 = u(X \cup Y \cup A)$,
  contradicting feasibility.

Two questions

1. To what extent does the theory of submodular functions more generally extend if we replace every graph in sight with a filter graph and every point mass in sight with an ultrafilter measure?
   My guess: all the main results should extend once reformulated correctly.
2. What if we consider submodular functions on standard measure spaces (e.g. $\mathbb{R}^n$) and consider continuous measures as mass assignments?
   My guess: there should be a nice theory here, including some max-flow theorems.