The Local Lemma in descriptive combinatorics: survey and recent developments

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joint with **Felix Weilacher** (CMU)

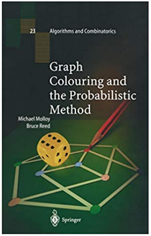
Georgia Institute of Technology

Caltech Logic Seminar

The Lovász Local Lemma

The **Lovász Local Lemma** (the **LLL**) is a powerful probabilistic tool.

- Introduced by Erdős and Lovász in ’75.
- Useful for proving existence results.
- Used throughout combinatorics.



- Recently found a number of applications in other areas (topological dynamics, ergodic theory, descriptive set theory).

I will survey some recent work on the LLL in the **Borel/continuous/measurable/Baire-measurable** context.

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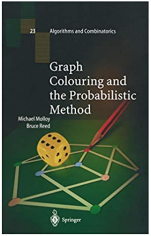

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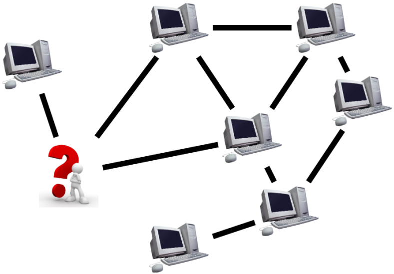
Deep connections to distributed algorithms.

Distributed algorithms

LOCAL model of distributed computation (Linial '92).

- Each vertex of G is an independent agent.
- Vertices can pass messages to their neighbors.
- Eventually, each vertex has to decide on its part of the solution.
- Complexity: the number of communication rounds.



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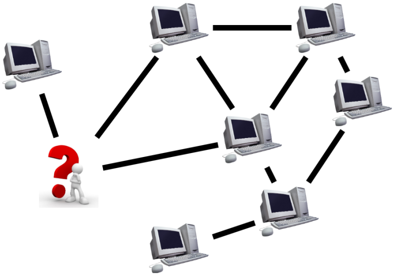

Deterministic and randomized versions.

Definitions and notation

A framework for the LLL: **constraint satisfaction problems**.

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A k-coloring of X is a function $f\colon X \to k$.

A constraint is a set B of functions $D \to k$, where D is a finite subset of X called the domain of B. Write $\mathrm{dom}(B) := D$.

A function $f\colon X \to k$ violates B if $f|_D \in B$. Otherwise, f satisfies B.

The functions in B are "bad" and we want to avoid them.

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Throughout we'll assume that $\sup_{B \in \mathscr{B}} |\mathrm{dom}(B)| < \infty$.

Definitions and notation

Let $\mathscr{B} \colon X \to^? k$ be a CSP. The probability of a constraint $B \in \mathscr{B}$ is

$$\mathbb{P}[B] := \frac{|B|}{k^{|\mathrm{dom}(B)|}} = \text{prob. } B \text{ is violated by a random coloring.}$$

The degree of a constraint $B \in \mathscr{B}$ is

$$\deg(B) := |\{B' \in \mathscr{B} \setminus \{B\} : \mathrm{dom}(B) \cap \mathrm{dom}(B') \neq \varnothing\}|.$$

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Theorem (Erdős–Lovász '75): the Lovász Local Lemma

If $e \cdot p(\mathscr{B}) \cdot (d(\mathscr{B}) + 1) \leqslant 1$, then $\mathscr{B}$ has a solution.

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More general versions also exist.

Example: coloring $\mathbb{R}$

Proposition

Let m, $k \geqslant 2$ be integers satisfying $m > 100k \log k$. Then, for any set S of m real numbers, there is a coloring $f : \mathbb{R} \to k$ such that each translation of S contains elements of all k colors.

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Proof sketch. Let S be a set of $m = \lceil 100k \log k \rceil$ real numbers.

For each $x \in \mathbb{R}$, let B_x be a constraint with domain $x + S$ consisting of all colorings $\varphi\colon (x+S) \to k$ that don't use all k colors.

Set $\mathscr{B} := \{B_x : x \in \mathbb{R}\}$. This is a CSP and we want a solution to $\mathscr{B}$.

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$$p(\mathscr{B}) \leqslant \frac{k(k-1)^m}{k^m} = k\left(1 - \frac{1}{k}\right)^m \leqslant k\,e^{-m/k} < k^{-99}.$$

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We have $e \cdot p(\mathscr{B}) \cdot (d(\mathscr{B}) + 1) < 1 \xrightarrow{\text{LLL}}$ there is a solution. ∎

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Note that there are uncountably many constraints. The proof uses a compactness argument.

Can we find a "nice" coloring $f : \mathbb{R} \to k$ as above?

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There is a Borel function $f : \mathbb{R} \to k$ as above.

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Can make f continuous away from a countable set of points.

The LLL in the Borel setting

Let X a Polish space.

Let $\mathscr{B} \colon X \to^{?} k$ a Borel CSP (as a subset of $[[X \times k]^{<\infty}]^{<\infty}$).

Can we use the LLL to find a Borel solution $f \colon X \to k$?

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A sinkless orientation of G is a solution to a CSP

$$\mathscr{B}_{\text{sinkless}} = \{B_x\}_{x \in V(G)} : E(G) \to^? 2.$$

Here B_x is the constraint with domain $\{e \in E(G) : e \text{ is incident to } x\}$ saying that at least one edge must be leaving x.

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Bad example (cont'd)**:**

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$$d(\mathscr{B}_{\text{sinkless}}) = d \quad \text{and} \quad p(\mathscr{B}_{\text{sinkless}}) = 2^{-d}.$$

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Certainly, $e \cdot p \cdot (d+1) = e2^{-d}(d+1) \ll 1$ for large d. But:

Theorem (Thornton '20)

For any $d \in \mathbb{N}$, there exists a d-regular Borel graph G without a Borel sinkless orientation.

Here and in what follows, Borel graphs are always on Polish spaces.

The LLL in the Borel setting

Theorem

Let $\mathscr{B}\colon X \to^? k$ be a Borel CSP on a Polish space X such that

$$p(\mathscr{B}) < 2^{-d(\mathscr{B})}.$$

Then $\mathscr{B}$ admits a Borel solution.

As the sinkless orientation example shows, $<$ can't be replaced by $\leqslant$.

Consequence of two facts:

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X is 0-dimensional + some natural topological assumptions on $\mathscr{B}$:
If $p(\mathscr{B}) < 2^{-d(\mathscr{B})}$, then $\mathscr{B}$ admits a continuous solution $f\colon X \to k$.

Another continuous version of the LLL

Given a CSP $\mathscr{B}$, define the vertex degree and the order

$$\mathsf{vdeg}(\mathscr{B}) := \max_{x \in X} |\{B \in \mathscr{B} : x \in \mathrm{dom}(B)\}|, \qquad \mathsf{ord}(\mathscr{B}) := \max_{B \in \mathscr{B}} |\mathrm{dom}(B)|.$$

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Theorem (AB '23)

Under some natural topological assumptions: If

$$p(\mathscr{B}) \cdot \mathsf{vdeg}(\mathscr{B})^{\mathsf{ord}(\mathscr{B})} < 1,$$

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Applications in topological dynamics (constructions of free subshifts with desirable properties).

Dependency graphs

Let $\mathscr{B} \colon X \to^? k$ be a CSP. The dependency graph of $\mathscr{B}$:

- vertices $\rightsquigarrow$ the constraints in $\mathscr{B}$
- edges $\rightsquigarrow$ pairs (B, B') with $\mathrm{dom}(B) \cap \mathrm{dom}(B') \neq \varnothing$

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Let $\mathscr{B} \colon X \to^? k$ be a Borel CSP on a Polish space X such that:

- $e \cdot p(\mathscr{B}) \cdot (d(\mathscr{B}) + 1) \leqslant 1$ (usual LLL assumption),
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Applications to polynomial growth graphs:

Theorem (AB–Yu)

Borel graphs of polynomial growth are hyperfinite.

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May 10: UCLA Logic Colloquium by Jing Yu about this!

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Continuous solution? We don't know! But:

Under some natural topological assumptions: Suppose that

- $e \cdot p(\mathscr{B}) \cdot (d(\mathscr{B}) + 1) \leqslant 1$ (usual LLL assumption),
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- the dependency graph of $\mathscr{B}$ is of sufficiently slow growth (polynomial is enough).

Then $\mathscr{B}$ admits a continuous solution.

The proof uses a deterministic distributed algorithm for the LLL due to ROZHOŇ–GHAFFARI '20.

The ROZHOŇ–GHAFFARI algorithm runs in time $O(\log^c n)$ for some constant c. To get a continuous solution need the growth rate of the dependency graph to be below $\exp(o(r^{1/c}))$. (Polynomial suffices.)

Measurable versions of the LLL

The LLL is a probabilistic statement, so perhaps it should interact well with measure theory.

Conjecture

Let $\mathscr{B} \colon X \to^{?} k$ be a Borel CSP on a Polish space X and let μ be a probability measure on X. Suppose that $e \cdot p(\mathscr{B}) \cdot (d(\mathscr{B}) + 1) \leqslant 1$ (usual LLL assumption). Then $\mathscr{B}$ admits a μ-measurable solution.

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Can get a Borel coloring that satisfies the constraints away from a set of measure $\leqslant \varepsilon$, for any given $\varepsilon > 0$.

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Same conjecture for Baire-measurable solutions.

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The proof relies on an efficient randomized algorithm for the LLL due to Fischer and Ghaffari '17 and is used to show:

Corollary (informal; AB)

efficient rand. distributed algorithms $\Longrightarrow$ measurable solutions

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Component-finite graphs

A graph G is component-finite if every component of G is finite.

Borel combinatorics trivialize on component-finite graphs:

Easy fact

Let $\mathscr{B}\colon X \to^{?} k$ be a Borel CSP on a Polish space X that has a solution. If the dependency graph of $\mathscr{B}$ is component-finite, then $\mathscr{B}$ has a Borel solution.

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For a graph G of maximum degree $d \geqslant 3$,

- $\chi(G) \leqslant d$ if G has no $(d+1)$-clique (Brooks '41);
- $\chi(G) = O(d/\log d)$ if G is triangle-free (Johansson' 96).

Asymptotic separation index

Let G be a locally finite Borel graph.

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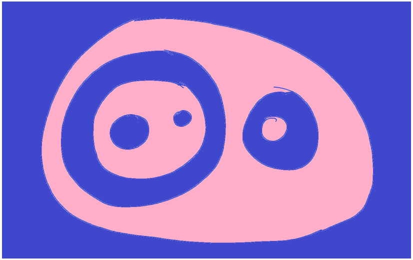
A locally finite Borel graph G has asymptotic separation index $\leqslant s$, in symbols $\mathrm{asi}(G) \leqslant s$, if for all $r \in \mathbb{N}$, G^r is a union of $s+1$ Borel component-finite induced subgraphs.



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Open question

Does $\mathrm{asi} < \infty$ imply $\mathrm{asi} \leqslant 1$?

Asymptotic separation index and the LLL

Theorem (AB–Weilacher)

Let $\mathscr{B}$ be a Borel CSP on a Polish space such that the dependency graph of $\mathscr{B}$ has $\mathsf{asi} \leqslant s$. If either

$$e^{s+1} \cdot p(\mathscr{B}) \cdot (d(\mathscr{B})+1)^{s+1} \leqslant 1 \qquad \text{or} \qquad 2^{15} \cdot p(\mathscr{B}) \cdot (d(\mathscr{B})+1)^{8} \leqslant 1,$$

then $\mathscr{B}$ admits a Borel solution.

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Corollary (AB–Weilacher)

Let $\mathscr{B}$ be a Borel CSP on a Polish space. Suppose that

$$e^{2} \cdot p(\mathscr{B}) \cdot (d(\mathscr{B})+1)^{2} \leqslant 1.$$

Then $\mathscr{B}$ has a Baire-measurable solution.

Also, if the dependency graph is hyperfinite, then $\mathscr{B}$ admits a μ-measurable solution for any probability measure μ.

Some consequences

As mentioned before:

Theorem (informal; AB '23)

efficient deterministic distributed algorithms $\Longrightarrow$ Borel & continuous solutions

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If G is a Borel graph of maximum degree $d \geqslant 3$ with $\mathsf{asi}(G) < \infty$, then:

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Summary

Does the LLL yield Borel/continuous solutions?

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- **YES** for Borel if
  - $p(d+1)^2 \leqslant e^{-2}$ for $\mathsf{asi} \leqslant 1$, or
  - $p(d+1)^8 \leqslant 2^{-15}$ for $\mathsf{asi} < \infty$.

Summary

Does the LLL yield measurable solutions?

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- **YES** with an ε error
- **YES** if $p(d+1)^8 \leqslant 2^{-15}$

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Does the LLL yield Baire-measurable solutions?

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Proof ideas

Let $\mathscr{B}\colon X \to^? k$ be a Borel CSP on a Polish space X.

Suppose that:

- the dependency graph G of $\mathscr{B}$ has $\mathsf{asi}(G) \leqslant s$;
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Keywords: method of conditional probabilities.

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Apply this with $C = e^{s+1}$, $N = s+1$.

Thank you!