Every CBER is smooth below the Carlson-Simpson generic partition

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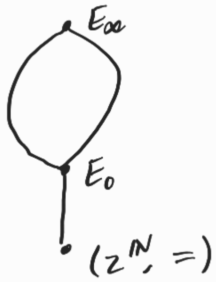
Recall:

1. E is a CBER on X if
   (a) $E \subseteq X \times X$ is a Borel equiv rel
   (b) every E-class is ctble
2. $(X,E) \leq_B (Y,F)$ if $\exists$ Borel $\varphi: X \to Y$ $\forall x_0, x_1 \in X$ $x_0 E x_1 \Leftrightarrow \varphi(x_0) F \varphi(x_1)$

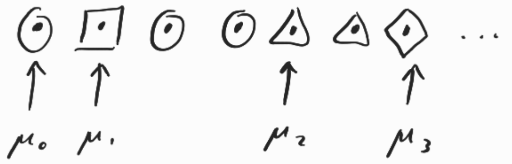


Q: How do CBERs behave on "large" sets?

Carlson-Simpson space

Def: $\mathcal{E}_\infty$ = all equiv rels A on $\mathbb{N}$ s.t. $\mathbb{N}/A$ is infinite
= partitions of $\mathbb{N}$ into infinitely many classes

$\mu(A) := \{\min [x]_A \mid x \in \mathbb{N}\}$
$= \{\mu_n(A) \mid n \in \mathbb{N}\}$ increasing enumeration



For $A, B \in \mathcal{E}_\infty$, $A \leq B$ if every A-class is a union of B-classes.

$r_n(A) := A \restriction \mu_n(A)$

$\mathcal{E}_\infty^{fin} := \{r_n(A) \mid A \in \mathcal{E}_\infty, n \in \mathbb{N}\}$

For $s, t \in \mathcal{E}_\infty^{fin}$, $s \leq t$ if dom s = dom t and s is coarser than t.

$s \sqsubseteq A \Leftrightarrow \exists n$ $s = r_n(A)$

$[s, A] := \{B \in \mathcal{E}_\infty \mid s \sqsubseteq B, B \leq A\}$

Polish topology: generated by all $[s, \mathbb{N}_{ER}]$

Ellentuck topology: generated by all $[s, A]$

Def: $\mathcal{X} \subseteq \mathcal{E}_\infty$ is Ramsey if $\forall [s, A] \neq 0$ $\exists B \in [s, A]$ s.t.

1. $[s, B] \subseteq \mathcal{X}$, or
2. $[s, B] \subseteq \mathcal{E}_\infty \setminus \mathcal{X}$.

$\mathcal{X}$ is Ramsey null if (2) always holds.

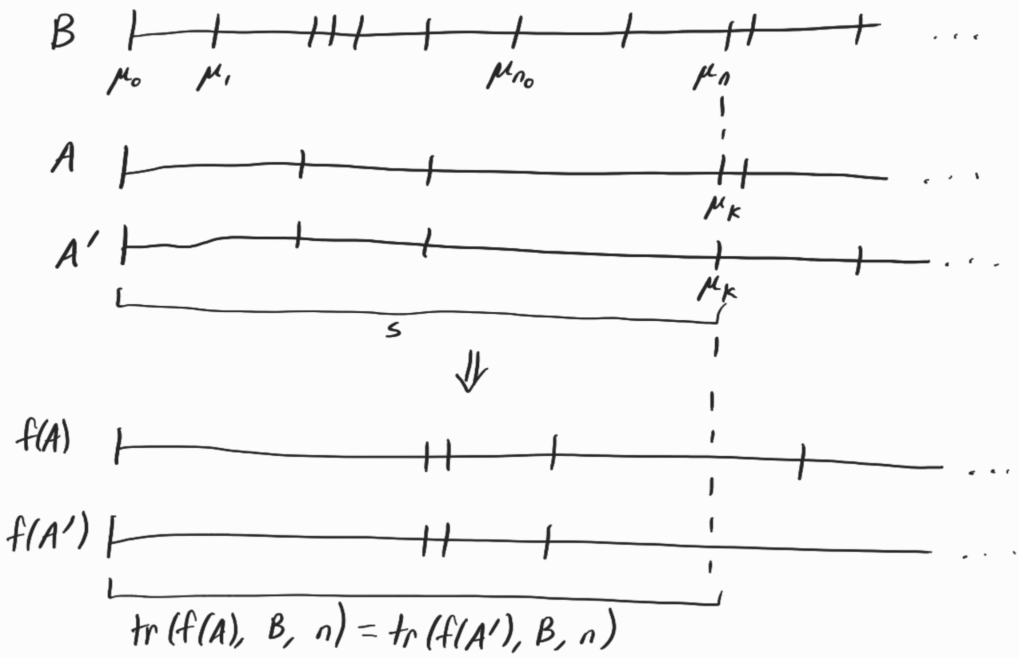
Thm (Carlson-Simpson): For $\mathcal{X} \subseteq \mathcal{E}_\infty$,

1. $\mathcal{X}$ has BP in the Ellentuck topology $\Longleftrightarrow \mathcal{X}$ is Ramsey.
2. $\mathcal{X}$ is meager " " " " $\Longleftrightarrow \mathcal{X}$ is Ramsey null.

Thm: Let E be a CBER on $\mathcal{E}_\infty$. Then $\exists$ Ramsey conull $\mathcal{D} \subseteq \mathcal{E}_\infty$ s.t. $\forall B \in \mathcal{D}$ $\forall A, A' \leq B$ $A E A' \Leftrightarrow A = A'$.

$tr(A, B, n) := A \restriction \mu_n(B)$

Def: $f: \mathcal{E}_\infty \to \mathcal{E}_\infty$ is tracial on $\mathcal{X} \subseteq \mathcal{E}_\infty$ if $\forall B \in \mathcal{X}$ $\exists n_0 \in \mathbb{N}$ $\forall n \geq n_0$ $\forall A, A' \leq B$, if $s = r_k(A) = r_k(A')$ for some k, s w/ $dom(s) = \mu_n(B)$, then $tr(f(A), B, n) = tr(f(A'), B, n)$.



Tracial Lem: Let $f: \mathcal{E}_\infty \to \mathcal{E}_\infty$ be Borel. Then $\exists$ Ramsey conull $\mathcal{C} \subseteq \mathcal{E}_\infty$ on which f is tracial.

Main Lem: Let $f: \mathcal{E}_\infty \to \mathcal{E}_\infty$ be Borel. Then $\exists$ Ramsey conull $\mathcal{D} \subseteq \mathcal{E}_\infty$ s.t. $\forall B \in \mathcal{D}$ $\forall A \leq B$ $f(A) \leq B \Rightarrow f(A) \leq A$.

Pf of Thm:

given CBER E on $\mathcal{E}_\infty$, fix Borel involutions $\{f_n \mid n \in \mathbb{N}\}$ s.t. $E = \bigcup_n f_n$

for each n, fix Ramsey conull $\mathcal{D}_n$ s.t. $\forall B \in \mathcal{D}_n$ $\forall A \leq B$ $f_n(A) \leq B \Rightarrow f_n(A) \leq A$

$\mathcal{D} := \bigcap_n \mathcal{D}_n$

suppose $B \in \mathcal{D}$, $A, A' \leq B$ s.t. $A E A'$

fix n s.t. $f_n(A) = A'$ and $f_n(A') = A$

$B \in \mathcal{D} \subseteq \mathcal{D}_n \Rightarrow A \leq A'$ and $A' \leq A$

$\Rightarrow A = A'$ $\square$

Pf of Main Lem:

consider any $[s, D] \neq \emptyset$

suffices to find $B \in [s, D]$ s.t. if $A \leq B$, then $f(A) \leq B \Rightarrow f(A) \leq A$

by Tracial Lem, fix $\mathcal{C} \subseteq \mathcal{E}_\infty$ Ramsey conull on which f is tracial

fix $B_0 \in \mathcal{C} \cap [s, D]$

let $n_0 \geq depth_{B_0}(s)$ witness traciality we will construct $\langle B_\ell \mid \ell \in \mathbb{N} \rangle$ s.t.

1. $B_{\ell+1} \in [r_{n_0+\ell}(B_\ell), B_\ell]$
2. if $A \le B_{\ell+1}$ and $\exists k$ $\mu_k(A) = \mu_{n_0+\ell+1}(B_{\ell+1})$, then
   (a) $tr(f(A), B_{\ell+1}, n_0+\ell+1) \le tr(A, B_{\ell+1}, n_0+\ell+1)$, or
   (b) $tr(f(A), B_{\ell+1}, n_0+\ell+1) \not\le r_{n_0+\ell+1}(B_{\ell+1})$

given such $\langle B_\ell \mid \ell \in \mathbb{N} \rangle$, let B be the limit in the Polish topology

for all ℓ, $B \in [r_{n_0+\ell}(B_\ell), B_\ell]$

$B \in [r_{n_0}(B_0), B_0]$

$\Rightarrow B \in [s, D]$

Clm: If $A \le B$ and $f(A) \le B$, then $f(A) \le A$.

Pf of Clm:

fix $\ell_0 < \ell_1 < \ell_2 < \dots$ s.t. $\forall i$ $\exists m_i$ $\mu_{m_i}(A) = \mu_{n_0+\ell_i+1}(B) = \mu_{n_0+\ell_i+1}(B_{\ell_i+1})$

either

(a) $tr(f(A), B_{\ell_i+1}, n_0+\ell_i+1) \le tr(A, B_{\ell_i+1}, n_0+\ell_i+1)$, or

(b) $tr(f(A), B_{\ell_i+1}, n_0+\ell_i+1) \not\le r_{n_0+\ell_i+1}(B_{\ell_i+1})$

(a) $\Leftrightarrow tr(f(A), B, n_0+\ell_i+1) \le tr(A, B, n_0+\ell_i+1)$

(b) $\Leftrightarrow tr(f(A), B, n_0+\ell_i+1) \not\le r_{n_0+\ell_i+1}(B)$

$tr(f(A), B, n_0+\ell_i+1) = f(A) \restriction \mu_{n_0+\ell_i+1}(B) \le r_{n_0+\ell_i+1}(B)$

$$\underbrace{tr(f(A), B, n_0+\ell_i+1)}_{= f(A) \restriction \mu_{n_0+\ell_i+1}(B)} \le \underbrace{tr(A, B, n_0+\ell_i+1)}_{= A \restriction \mu_{n_0+\ell_i+1}(B)}$$

$\Rightarrow f(A) \le A$ $\square$ Clm

$Q(n) :=$ set of partitions of n

$Q^m(n) := \{ s \in Q(n) \mid s$ has exactly m classes, $i < j < m \Rightarrow \neg i \, s \, j \}$

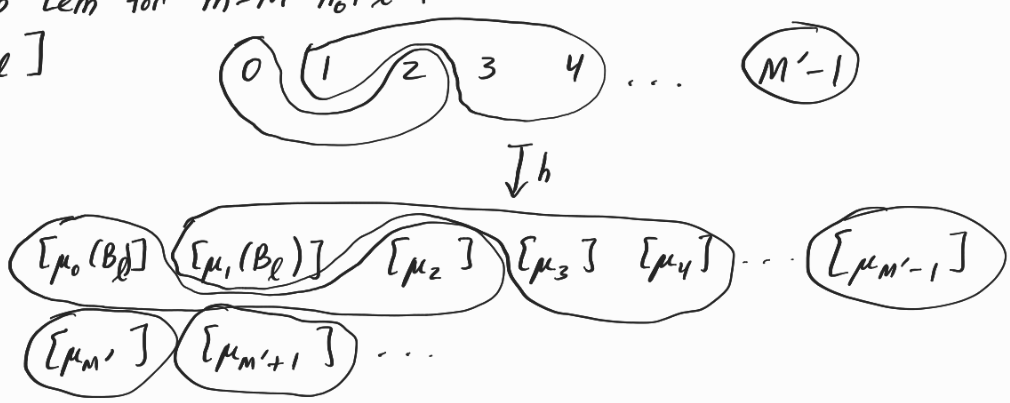
Combinatorial Lem: For all $m, M \in \mathbb{N}$, $\exists$ $M' \ge m, M$ s.t. $\forall e : Q(M') \to Q(M')$ $\exists$ $s \in Q^m(M')$ $\forall$ $t \le s$ $e(t) \le t$ or $e(t) \not\le s$.

Constructing $\langle B_\ell \mid \ell \in \mathbb{N} \rangle$

suppose we have B_ℓ

fix M' as in Comb Lem for $m = M = n_0 + \ell + 1$

$h : Q(M') \to [\emptyset, B_\ell]$



$g : \mathcal{E}_\infty \longrightarrow Q(M')$

if $C \upharpoonright \mu_{M'}(B_\ell) \leq r_{M'}(B_\ell)$, let $i \; g(C) \; j \iff \mu_i(B_\ell) \subset \mu_j(B_\ell)$

o/w, let $i \; g(C) \; j \iff i = j$

observe: $C \upharpoonright \mu_{M'}(B_\ell) \leq r_{M'}(B_\ell) \Rightarrow h(g(C)) \upharpoonright \mu_{M'}(B_\ell) = C \upharpoonright \mu_{M'}(B_\ell)$

define $e : Q(M') \longrightarrow Q(M')$ by $e(t) := g(f(h(t)))$

fix $s' \in Q^{n_0+\ell+1}(M')$ s.t. $\forall\, t \leq s'$ $e(t) \leq t$ or $e(t) \not\leq s'$

$B_{\ell+1} := h(s') \in [\emptyset, B_\ell]$

$s' \in Q^{n_0+\ell+1}(M') \Rightarrow r_{n_0+\ell}(B_\ell) = r_{n_0+\ell+1}(B_{\ell+1})$

$\Rightarrow B_{\ell+1} \in [r_{n_0+\ell}(B_\ell), B_\ell]$

2. if $A \leq B_{\ell+1}$ and $\exists\, k$ s.t. $\mu_k(A) = \overset{\overset{\mu_n(B_0)}{\|}}{\mu_{n_0+\ell+1}}(B_{\ell+1})\overset{\overset{\mu_{M'}(B_\ell)}{\|}}{}$, then

   (a) $tr(f(A), B_{\ell+1}, n_0+\ell+1) \leq tr(A, B_{\ell+1}, n_0+\ell+1)$ or

   (b) $tr(f(A), B_{\ell+1}, n_0+\ell+1) \not\leq r_{n_0+\ell+1}(B_{\ell+1})$

"$\square$"