Borel asymptotic dimension and hyperfiniteness

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Part I

Overview

Many interesting Borel equivalence relations may be realized as "finite distance" equivalence relations arising from appropriate Borel extended metrics.

We define and investigate a notion of *asymptotic dimension* of such metrics, and connect this notion to hyperfiniteness of their corresponding equivalence relations.

Our proofs will be by cartoon.

This is joint work with Steve Jackson, Andrew Marks, Brandon Seward, and Robin Tucker-Drob.

Part II

Borel extended metrics

Definition

Throughout, X will denote some *standard Borel space*, i.e., a set equipped with a σ-algebra of Borel sets arising from some Polish topology.

Definition

A *Borel extended metric* on X is a Borel function $\rho\colon X^2 \to [0, \infty]$ satisfying the usual axioms of an extended metric.

Definition

Such a metric is *proper* if every ball of finite radius is finite.

Definition

Given such a Borel extended metric (X, ρ), with each $r < \infty$ we may associate a Borel graph $\mathcal{G}_r$ on X by declaring

$$x \, \mathcal{G}_r \, y \iff x \neq y \text{ and } \rho(x, y) < r.$$

When ρ is proper, each $\mathcal{G}_r$ is locally finite.

Definition

With (X, ρ) we may also associate a Borel equivalence relation E_ρ by declaring

$$x \; E_\rho \; y \iff \rho(x, y) < \infty.$$

When ρ is proper, E_ρ is a CBER.

Example A

Suppose that G is a locally finite Borel graph on X. Then the corresponding graph metric ρ is a proper Borel extended metric. In this case, E_ρ is the connectedness relation of G.

Example B

Suppose that Γ is a countable group and that d is a proper right-invariant metric on Γ. With any free Borel action $\Gamma \curvearrowright X$ we may associate a proper Borel extended metric ρ by

$$\rho(x,y) = \begin{cases} d(e,g) & \text{if } y = g \cdot x \\ \infty & \text{otherwise.} \end{cases}$$

In this case, E_ρ is the orbit equivalence relation of $\Gamma \curvearrowright X$.

Question

How do properties of (X, ρ) relate to properties of E_ρ?

We are particularly interested in detecting hyperfiniteness of E_ρ.

Definition

A CBER E is *hyperfinite* if it is an increasing union of FBERs. That is, if there exist (class-)finite Borel equivalence relations $F^0 \subseteq F^1 \subseteq \cdots$ with $E = \bigcup_n F^n$.

Question (Weiss)

Is every orbit equivalence relation of a Borel action of a countable amenable group hyperfinite?

Part III

Borel asymptotic dimension

From now on, (X, ρ) will be some proper Borel extended metric.

Definition

A family $\mathcal{C} \subseteq \mathcal{P}(X)$ is *uniformly* $(\rho$-$)$*bounded* if there is $R < \infty$ such that for all $C \in \mathcal{C}$, $\operatorname{diam}_\rho(C) < R$.

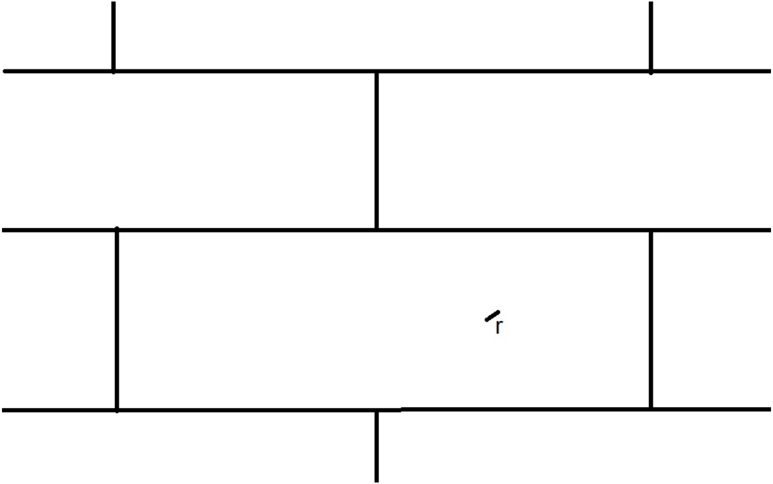
Definition

Given $s \in \mathbb{N}$, we say $\operatorname{asdim}_{\mathrm{B}}(X, \rho) \leq s$ if for all $r < \infty$ there is a Borel partition $X = W_0 \sqcup W_1 \sqcup \cdots \sqcup W_s$ so that for all i, the connected components of $\mathcal{G}_r \upharpoonright W_i$ are uniformly bounded.

Remark

There are several equivalent definitions, but I like this one because it reminds me of graph coloring. An example will help clarify why it is a notion of "dimension."

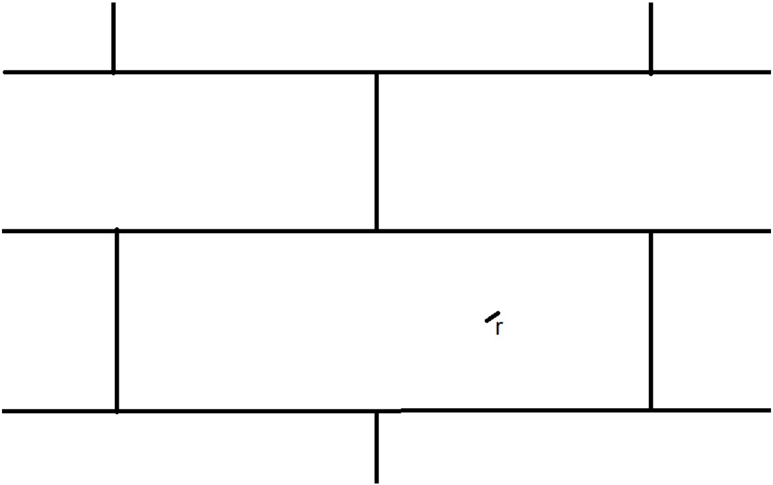
The usual (or any other) proper metric on $\mathbb{Z}^2$ should be "2-dimensional." One way of formalizing this, inspired by covering dimension, is by tiling by large bricks.



III. Borel asymptotic dimension

Example: $\mathbb{Z}^2$

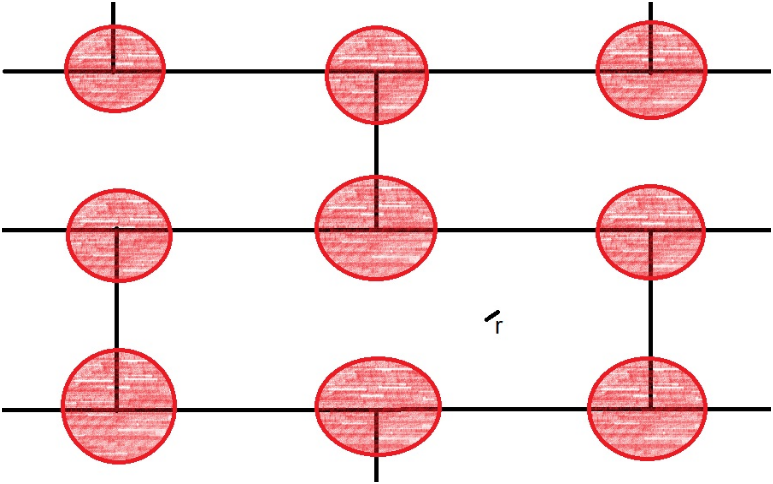
We use this pattern to witness $\mathrm{asdim}_{\mathrm{B}}(\mathbb{Z}^2) \leq 2$ by coloring points in turn. $X = \cdots$



III. Borel asymptotic dimension

Example: $\mathbb{Z}^2$

We use this pattern to witness $\text{asdim}_B(\mathbb{Z}^2) \leq 2$ by coloring points in turn. $X = {\color{red}W_0} \sqcup \cdots$



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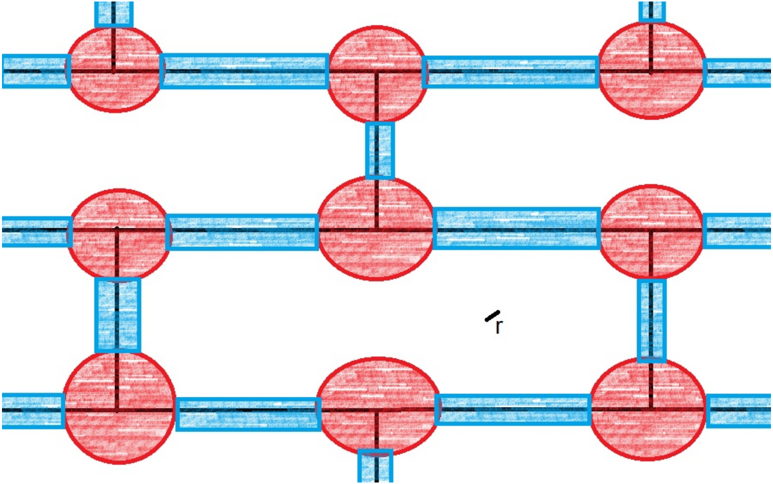

III. Borel asymptotic dimension

Example: $\mathbb{Z}^2$

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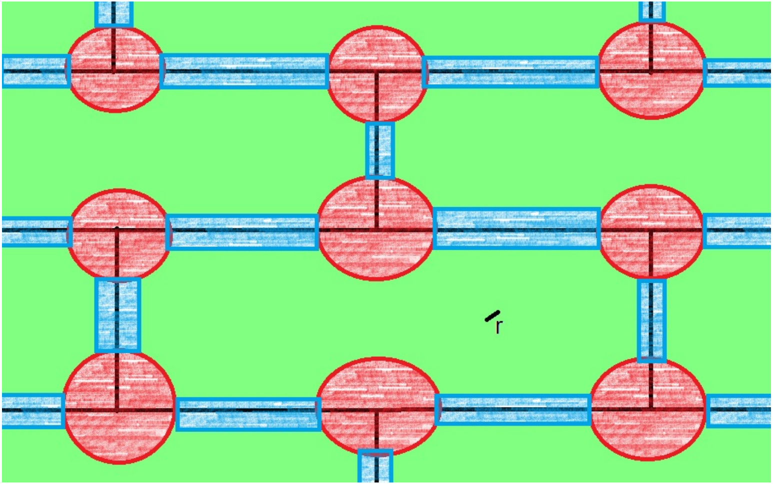
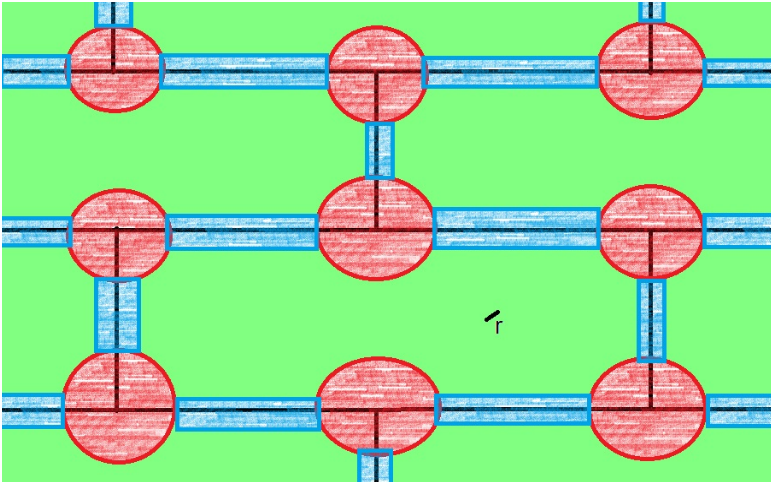


III. Borel asymptotic dimension

Example: $\mathbb{Z}^2$

This argument generalizes to convert between various different definitions of $\mathrm{asdim}_{\mathrm{B}}(X, \rho) \leq s$.



Definition

Given $s \in \mathbb{N}$, we say $\mathrm{asdim}_{\mathrm{B}}(X, \rho) \leq s$ if for all $r < \infty$ there is a Borel partition $X = W_0 \sqcup W_1 \sqcup \cdots \sqcup W_s$ so that for all i, the connected components of $\mathcal{G}_r \restriction W_i$ are **uniformly** bounded.

Remark

If we weaken the above definition to instead require only that the connected components of $\mathcal{G}_r \restriction W_i$ are ρ-bounded (i.e., finite) we obtain the definition of $\mathrm{asi}_{\mathrm{B}}(X, \rho) \leq s$.

Remark

This notion of asymptotic separation index is reminiscent of "toast" constructions from descriptive combinatorics, and finite asi is enough to run some arguments from that domain. These ideas are further developed via the ASI algorithms of Qian-Weilacher and the ASI local lemma of Bernshteyn-Weilacher.

Theorem

Suppose that $s \in \mathbb{N}$ and that (X, ρ) is a proper Borel extended metric with $\mathrm{asdim}_{\mathrm{B}}(X, \rho) \leq s$. Then E_ρ is hyperfinite.

Sketch of the proof

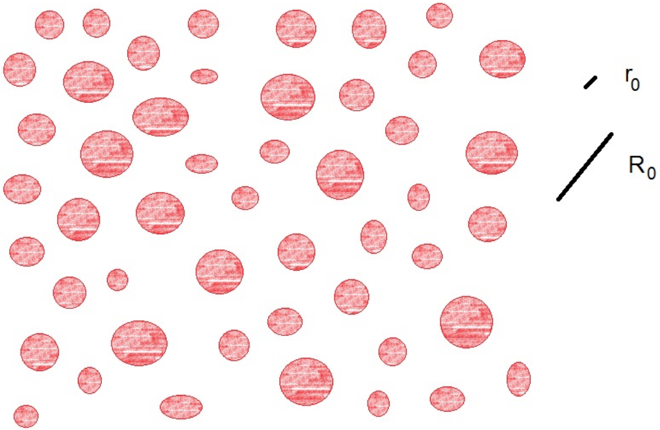
By assumption, we know for each $r < \infty$ we have a Borel partition $X = W_0^r \sqcup \cdots \sqcup W_s^r$ so that the connected components of $\mathcal{G}_r \restriction W_i^r$ are uniformly bounded.

The main idea is to "switch indices" and fix $i \leq s$ while examining how the sequence W_i^r behaves as r grows rapidly. We use these to define for each i increasing (partial) FBERS $F_i^0 \subseteq F_i^1 \subseteq \cdots$ which will help establish hyperfiniteness of E_ρ.

We illustrate this process for $i = 0$, which we view as red.

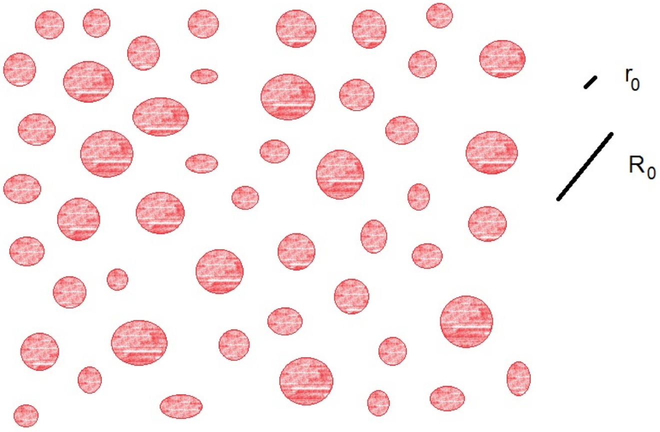
Sketch of the proof, cont.

Stage 0: $r_0 = 1$. Obtain W^{r_0} that looks like this in each E_ρ-class:



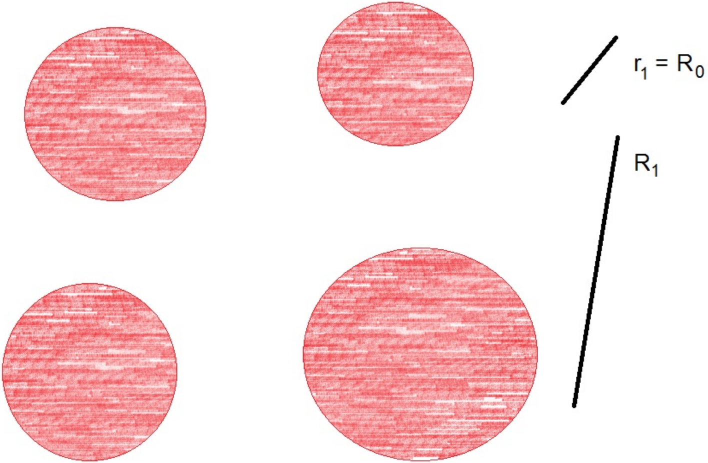
Sketch of the proof, cont.

Stage 0: $r_0 = 1$. Each red blob will form an F^0-class. These are finite, and in fact uniformly ρ-bounded by some R_0.



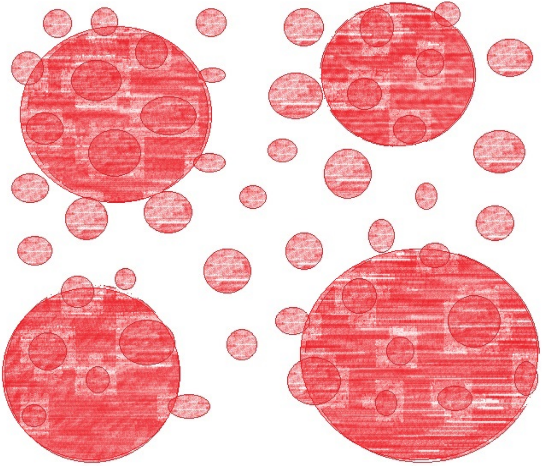
Sketch of the proof, cont.

Stage 1: $r_1 = R_0$. Obtain W^{r_1} that looks like this in each E_ρ-class:



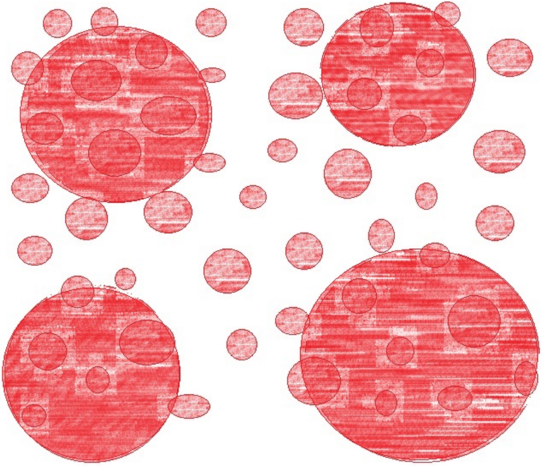
Sketch of the proof, cont.

Stage 1: $r_1 = R_0$. Superimpose with the W^{r_0} picture, obtaining:



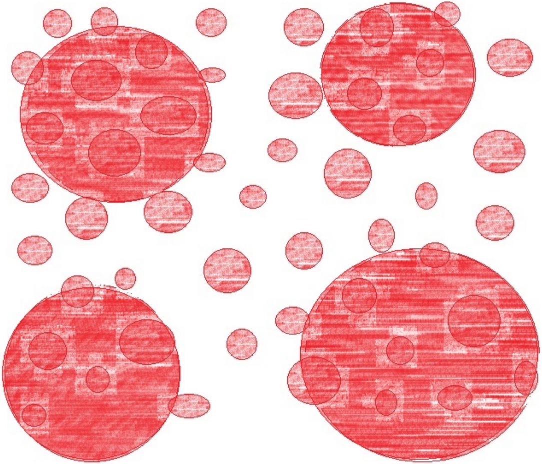
Sketch of the proof, cont.

Stage 1: $r_1 = R_0$. By construction, each F^0-class meets at most one $\mathcal{G}_{r_1} \upharpoonright W^{r_1}$-component.



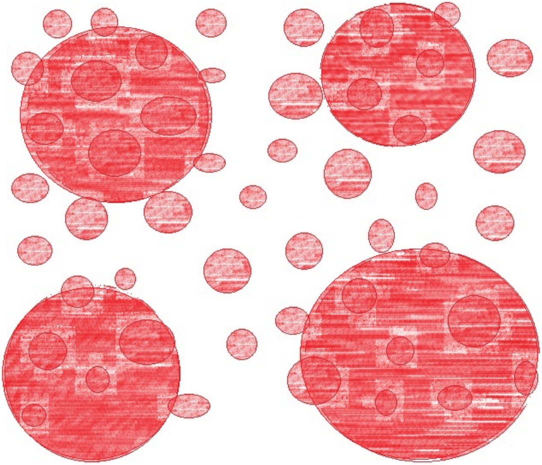
Sketch of the proof, cont.

Stage 1: $r_1 = R_0$. These contiguous regions will form our F^1-classes.



Sketch of the proof, cont.

Stage 1: $r_1 = R_0$. Observe that $F^0 \subseteq F^1$, and that each F^1-class has diameter at most $2R_0 + R_1$ (hence is finite).



Sketch of the proof, cont.

Stage 2: $r_2 = 2R_0 + R_1$. Each F^1-class meets at most one $\mathcal{G}_{r_2} \upharpoonright W^{r_2}$-component. The contiguous regions will form our F^2-classes, of diameter uniformly bounded by $2(2R_0 + R_1) + R_2$.

Stage 3: $r_3 = 2(2R_0 + R_1) + R_2 \ldots$

etc.

Sketch of the proof, cont.

- Proceeding in this fashion, we build for each $i \leq s$ (partial) FBERs $F_i^0 \subseteq F_i^1 \subseteq \cdots$, and thus hyperfinite (partial) equivalence relations $E_i = \bigcup_n F_i^n$.
- Using the fact that s is finite, a pigeonhole argument (essentially) ensures for each $i \leq s$ that each E_ρ-class contains at most one E_i-class.
- This grants the desired hyperfiniteness of E_ρ. $\square$

Part IV

Group actions

Application

Motivated by Weiss' question, the previous theorem grants a sufficient condition for a free Borel action $\Gamma \curvearrowright X$ to have hyperfinite orbit equivalence relation.

Remark

Recall, via Example B earlier, we have a mechanism for pushing a proper right-invariant metric on countable Γ forward to a proper Borel extended metric on X.

Remark

It is often convenient to use the following reformulation to avoid the hassle of fixing a right-invariant proper metric on Γ.

Reformulation

Any proper Borel extended metric ρ on X arising from a free Borel action $\Gamma \curvearrowright X$ has $\mathrm{asdim}_{\mathrm{B}}(X, \rho) \leq s$ if and only if

- for each finite $A \subseteq \Gamma$
- there is finite $B \subseteq \Gamma$ and Borel partition $X = W_0 \sqcup \cdots \sqcup W_s$
- so that for all $i \leq s$ and $x \in X$, $(\mathcal{G}_A \restriction W_i) \cdot x \subseteq B \cdot x$.

Definition

Here, $\mathcal{G}_A$ is the graph with edges $(x, a \cdot x)$ for $a \in A$.

Definition

Also, $\mathcal{G} \cdot x$ denotes the $\mathcal{G}$-connected component of x.

Definition

And $B \cdot x = \{b \cdot x : b \in B\}$.

Example

Any free Borel action of a locally finite group has $\operatorname{asdim}_{\mathrm{B}} = 0$, since for all finite A we have $\mathcal{G}_A \cdot x \subseteq \langle A \rangle \cdot x$.

Example (Jackson-Kechris-Louveau)

Any free Borel action of a finitely generated group of polynomial growth has finite $\operatorname{asdim}_{\mathrm{B}}$.

Example

Any free Borel action of the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ has $\operatorname{asdim}_{\mathrm{B}} = 1$.

Example

Any free Borel action of a polycyclic group has finite $\operatorname{asdim}_{\mathrm{B}}$.

So we obtain

Theorem

All of the actions on the previous page have hyperfinite orbit equivalence relations.

Remark

Dealing with non-free actions is a huge hassle, but it seems likely that they also have hyperfinite orbit equivalence relations.

Remark

For example, some general results of Schneider-Seward in conjunction with the asymptotic dimension machinery grant this for non-free Borel actions of polycyclic groups.

Part V

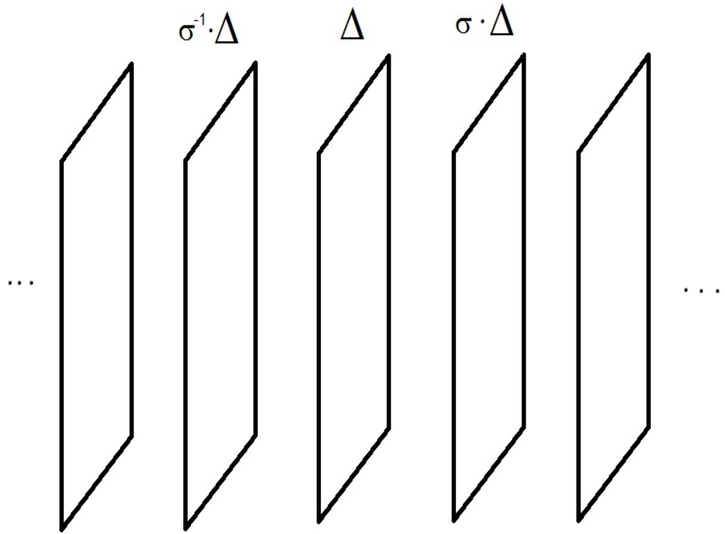
More on the lamplighter

We close with some cartoons indicating the overall structure of the argument that free Borel actions of the lamplighter have Borel asymptotic dimension 1.

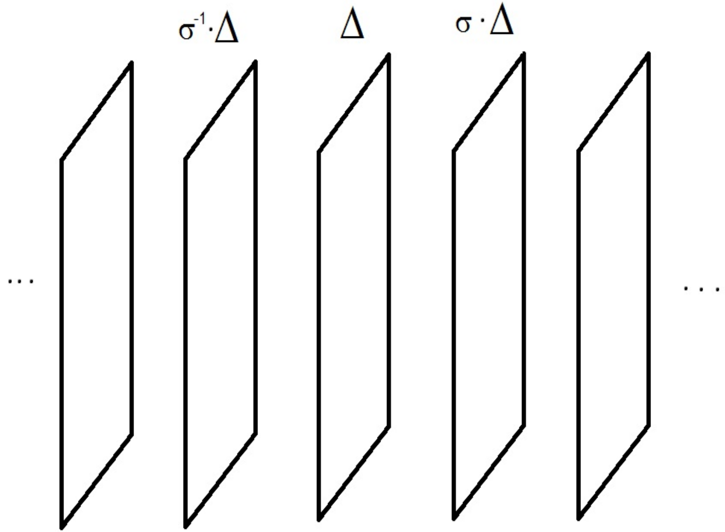
Definition

Put $\Delta = \bigoplus_{k \in \mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$, and let $\Gamma = \Delta \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts by the shift σ on the indices.

So the group looks something like this:



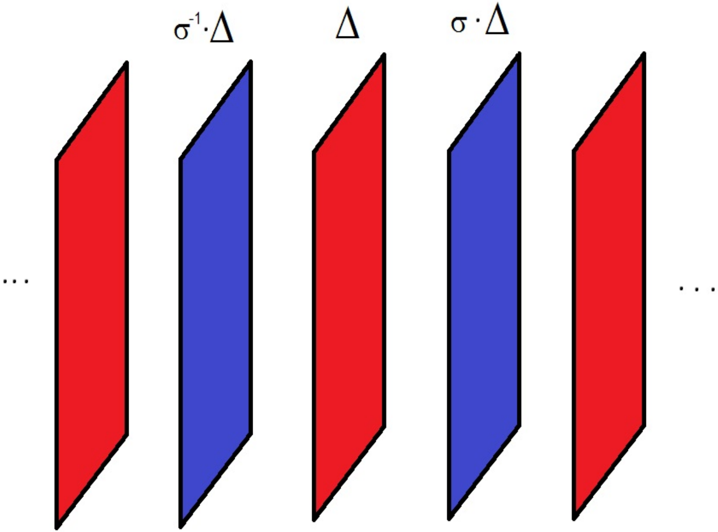
To witness (classical) asdim 1 with $A = A_\Delta \cup \{\sigma\}$:



V. More on the lamplighter

$\mathrm{asdim}_B = 1$

To witness (classical) asdim 1 with $A = A_\Delta \cup \{\sigma\}$:

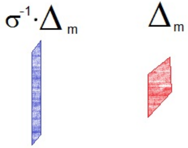


Remark

This type of approach can't work in the Borel context because there is generally no way of "aligning" the Δ-orbits like this in a Borel fashion.

Remark

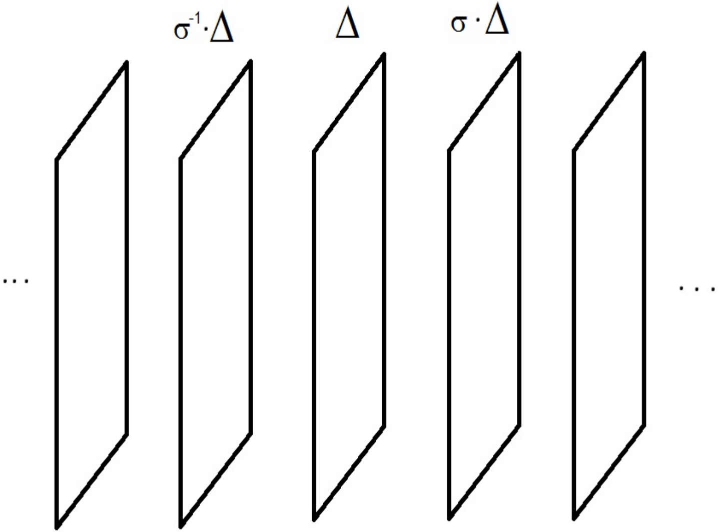
The work-around is to follow a "local algorithm," instead trying to color in stages by widgets:



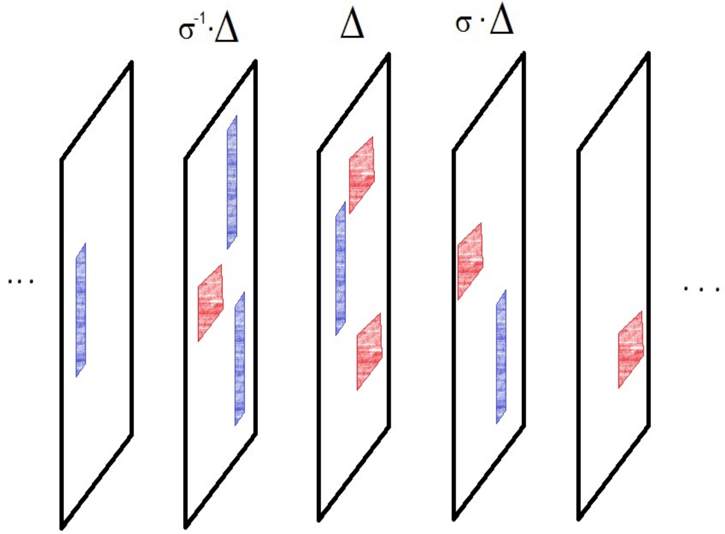
V. More on the lamplighter

$\mathrm{asdim}_B = 1$

The algorithm runs something like this:



The algorithm runs something like this:



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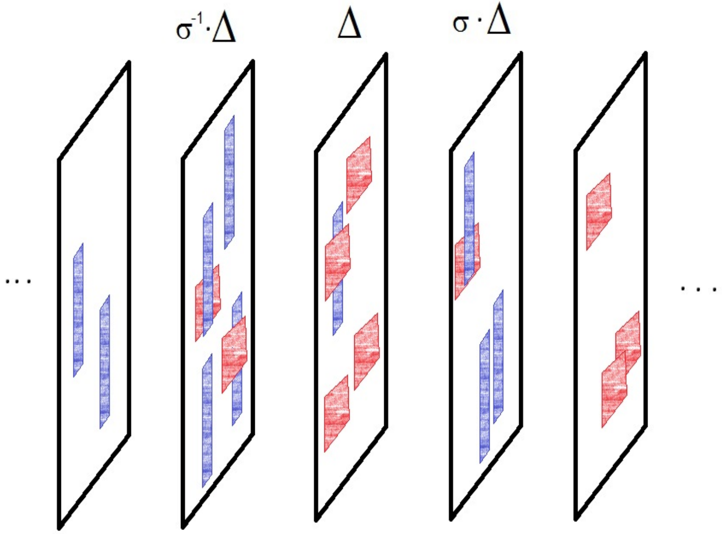
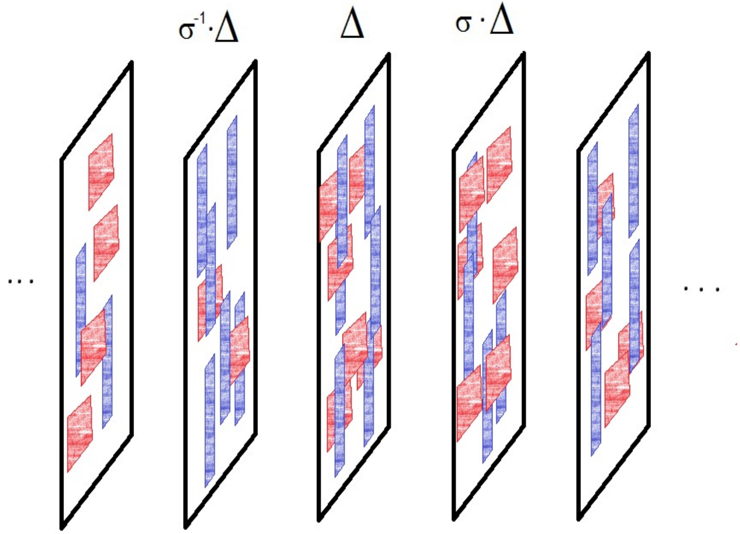


The algorithm runs something like this:



Remark

There is a scale-invariant widget conflict graph whose Borel chromatic number (at most 5, in this case) bounds the number of stages of this algorithm.

Remark

Widgets added at the same stage are disjoint, and widgets added at earlier stages take precedence over those added at later stages (unlike what I drew).

Remark

By choosing the original $\Delta_m \subseteq \Delta$ big enough, the resulting coloring will witness $\mathrm{asdim}_\mathrm{B} \leq 1$ for the given $A = A_\Delta \cup \{\sigma\}$.

Part VI

Thanks!