Groups without unitary representations, submeasures, and the escape property

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Supported by NSF grant DMS-2246873

April 2024

Joint work with **F. Martin Schneider**

Submeasures

$\mathcal{A}$ a boolean algebra (of sets)

A **submeasure** ϕ on $\mathcal{A}$ is a "norm" on $\mathcal{A}$ compatible with the boolean structure.

$\phi\colon \mathcal{A} \to \mathbb{R}$ such that, for $A, B \in \mathcal{A}$,

— $A \subseteq B \Rightarrow \phi(A) \leq \phi(B)$;

— $\phi(A \cup B) \leq \phi(A) + \phi(B)$;

— $\phi(A) \geq 0$ and $\phi(\emptyset) = 0$.

ϕ induces a (semi-)**metric** on $\mathcal{A}$

$$\mathrm{dist}_\phi(A, B) = \phi(A \triangle B)$$

A topological group associated with a submeasure ϕ

ϕ induces a **topological group structure** on $\mathcal{A}$

the group operation $= \triangle$ (symmetric difference)

the group topology $=$ given by dist_ϕ

Notation for the topological group: D_ϕ

Two types of submeasures

μ is a **measure** if μ is a **submeasure** and, for all $A, B \in \mathcal{A}$,

$$A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B).$$

ϕ is a **pathological submeasure** if ϕ is a **submeasure** and, for each **measure** $\mu \leq \phi$, we have $\mu = 0$.

Pathological submeasures were discovered a number of times in various mathematical contexts:

— graph theory, **Erdös–Hajnal**, 1967;

— Hausdorff measures, 1969;

— topological dynamics, 1975;

— ideals of subsets of $\mathbb{N}$, 1991.

Herer–Christensen: A **generic** submeasure is pathological.

$L^0(\phi, G)$ groups

G a topological group, ϕ a submeasure on $\mathcal{A}$

$$S(\phi, G) = \text{ } \mathcal{A}\text{-measurable step functions with values in } G$$

Group operation on $S(\phi, G)$ = pointwise multiplication

Group topology on $S(\phi, G)$ is generated by identity neighborhoods

$$\{a \in S(\phi, G) \mid \phi\big(a^{-1}(G \setminus U)\big) < \epsilon\}$$

for $\epsilon > 0$ and open $1 \in U \subseteq G$.

$\mathrm{L}^0(\phi, \mathrm{G}) =$ the (Raikov) completion of $S(\phi, G)$.

Example

μ a σ-additive (finite) measure on a σ-algebra

G a Polish group

Then

$$L^0(\mu, G) = \ \mu\text{-classes of } \mu\text{-measurable } G\text{-valued functions}$$

taken with convergence in measure μ.

A submeasure ϕ is a functor on the category of topological groups:

$$G \Longrightarrow L^0(\phi, G)$$

If ϕ is pathological, then $L^0(\phi, G)$ has intriguing properties, proofs of which involve unexpected methods.

Exotic groups

$\mathcal{U}(H) =$ **unitary operators** on a complex Hilbert space H

$\mathcal{U}(H)$ with the **strong operator topology** is a topological group

A topological group G is exotic if all continuous homomorphisms $G \to \mathcal{U}(H)$ (unitary representations) are **trivial**.

A connection with extreme amenability

G is **extremely amenable** if each continuous action of G on a compact space has a global fixed-point.

G **exotic** and **amenable** $\Rightarrow$ G **extremely amenable**.

The first examples of **exotic groups** and of **extremely amenable groups** were

Herer–Christensen: $L^0(\phi, \mathbb{R})$ with ϕ a pathological submeasure

Later, other examples of exotic groups were found by

Megrelishvili, **Banaszczyk**, **Carderi–Thom**.

Dynamics and representations of $L^0(\phi, G)$ Main results

Questions of interest

ϕ a pathological submeasure

Exoticness:
Are all unitary representations $L^0(\phi, G) \to \mathcal{U}(H)$ trivial?

Extreme amenability:
Do all continuous actions of $L^0(\phi, G)$ on compact spaces have fixed points?

Earlier results

Herer–Christensen, 1975: $L^0(\phi, \mathbb{R})$ is exotic and, therefore, extremely amenable

Farah–S., 2008: $L^0(\phi, G)$ is extremely amenable if G second countable, **compact**, and **nilpotent**

Sabok, 2012: $L^0(\phi, G)$ is extremely amenable if G second countable, **locally compact**, and **abelian**

Schneider–S., 2021: $L^0(\phi, G)$ is extremely amenable if G **amenable**

Theorem (Schneider–S.)

Let ϕ be a pathological submeasure.
Then $L^0(\phi, G)$ is exotic for each topological group G.

The theorem strengthens the results on the previous slide.

D_ϕ = the algebra $\mathcal{A}$ with the metric dist_ϕ induced by ϕ and with $\triangle$ as the group operation

Corollary (Schneider–S.)

The following conditions are equivalent.

- *ϕ is a pathological submeasure.*
- *$L^0(\phi, G)$ is exotic for some non-exotic G.*
- *D_ϕ is exotic.*
- *Every continuous homomorphisms $D_\phi \to D_\mu$, for a measure μ, is trivial.*

A theorem on triviality of $L^0 \to L^0$ homomorphisms

Theorem (Schneider–S.)

ϕ a pathological submeasure, G a topological group
μ a measure, H a topological group with the **escape property**
Then each continuous homomorphism

$$L^0(\phi, G) \to L^0(\mu, H)$$

is trivial.

The escape property

$f\colon G \to \mathbb{R}$ is a **length function** if it is continuous and, for all $x, y \in G$,

- $f(xy) \leq f(x) + f(y)$;
- $f(x^{-1}) = f(x)$;
- $f(x) \geq 0$ and $f(1) = 0$.

A general observation

G a topological group $\Rightarrow$

$$\{f^{-1}([0,1)) \mid f \text{ a length function on } G\}$$

is a neighborhood basis at 1.

$1 \in U \subseteq H$ open

$$\frac{1}{n}U = \{g \in H \mid g, g^2, \ldots, g^n \in U\}.$$

Note

$$1 \in \frac{1}{n+1}U \subseteq \frac{1}{n}U.$$

$f \colon H \to \mathbb{R}$ is an **escape function** on H if

- f is a length function and
- there exists $1 \in U \subseteq H$ open such that, for each $\epsilon > 0$, there exists n with

$$\frac{1}{n}U \subseteq f^{-1}([0, \epsilon)).$$

H a topological group

H has the **escape property** if

$$\{f^{-1}([0,1)) \mid f \text{ an escape function on } H\}$$

is a neighborhood basis at 1.

Groups with the escape property

- Banach-Lie groups (essentially due to Enflo)
- locally compact groups
- non-archimedean topological groups
- groups of isometries of locally compact separable metric spaces

Closure properties

- subgroups
- arbitrary products
- $\Sigma \ltimes G^I$ where Σ is a group of permutations of I and G has the escape property

An example of a group without the escape property

If ϕ a diffuse submeasure, K a topological group,
then $L^0(\phi, K)$ **does not** have the escape property,
in fact, all escape functions on it are constantly 0.

A comment on the proofs of the main results

A a countable **amenable** group, μ σ-additive

Consider $\pi\colon A \to L^0(\mu, \mathbb{R})$ such that, for $a, b \in A$,

- $\pi(ab) \leq \pi(a) + \pi(b)$;
- $\pi(a^{-1}) = \pi(a)$;
- $\pi(a) \geq 0$ and $\pi(1) = 0$.

Then, for all $r > 0$,

$$\mu\Big(\bigvee_{a \in A} \pi(a)^{-1}(r, \infty)\Big) \leq 4 \sup_{a \in A} \mu\big(\pi(a)^{-1}(r, \infty)\big).$$

Questions

Is $L^0(\phi, G)$ **extremely amenable** if G is a topological group and ϕ is a pathological submeasure?

Is $L^0(\phi, \mathbb{F}_2)$ **extremely amenable** if ϕ is a pathological submeasure?