Equivalence relations classifiable by Polish abelian groups

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Group actions

A group action $H \curvearrowright X$ induces an **orbit equivalence relation** on X: equivalence classes = orbits.

Important actions:

- Irrational rotation: $\mathbb{Z} \curvearrowright S^1$.
- Bernoulli shift: $\mathbb{Z} \curvearrowright 2^{\mathbb{Z}}$.
- Bernoulli shift: $F_2 \curvearrowright 2^{F_2}$.
- Graph isomorphism: $S_\infty \curvearrowright \{\text{graphs with vertex set } \mathbb{N}\}$.
- Cosets of a subgroup: $\ell^1 \curvearrowright \mathbb{R}^{\mathbb{N}}$.

General principle:

$$\text{group } H \text{ is nice} \implies \text{orbit equivalence relations of } H \text{ are nice}$$

Nice groups: abelian, amenable, TSI, non-archimedean...

Nice eq rels: smooth, hypersmooth, essentially countable, classifiable by countable structures...

Today's conventions (the zeitgeist)

All spaces today will be **standard Borel spaces**.

All equivalence relations today will be on such spaces, and they will be **Borel equivalence relations (BER)**.

Given spaces X and Y equipped with equivalence relations, a function $X \to Y$ is a **reduction** if for all $x, x' \in X$, we have that x and x' are equivalent (in X) iff $f(x)$ and $f(x')$ are equivalent (in Y).

All reductions today will be **Borel reductions**.

All groups will be **Polish groups**.

All actions today will be **Borel actions**.

Most equivalence relations today: orbit equivalence relation of $H \curvearrowright X$, where

- H is a Polish group
- X is a standard Borel space
- The action $H \curvearrowright X$ is Borel.

A **countable** Borel equivalence relation (CBER) is the orbit eq rel of an action of a countable group.

Classifying equivalence relations by a group

E: Borel eq rel.

H: Polish group.

E is **classifiable by** H if E reduces to an orbit eq rel of H.

Many results about BERs can be stated in this language.

1. The orbit eq rel of $F_2 \curvearrowright 2^{F_2}$ is not classifiable by $\mathbb{Z}$.
2. Every orbit eq rel of an action of a locally compact group is classifiable by a countable group.
3. No orbit eq rel of a turbulent action is classifiable by S_∞.

Classifiable by a countable group $=$ **essentially countable**.

Classifiable by $\mathbb{Z}$ $=$ **essentially hyperfinite**.

Hyperfiniteness

Classifiable by $\mathbb{Z}$ = **essentially hyperfinite**.
Gao-Jackson proved

$$\text{classifiable by a countable abelian group} \implies \text{essentially hyperfinite}$$

Hard conjecture: "abelian" can be replaced with "amenable".
How about **uncountable** abelian groups?

$$\text{classifiable by an abelian group} \implies \text{essentially hyperfinite??}$$

Cotton proved

$$\text{classifiable by } \textbf{locally compact} \text{ abelian} \implies \text{essentially hyperfinite}$$

Can't hold in full generality:
The natural action $\ell^1 \curvearrowright \mathbb{R}^{\mathbb{N}}$ is not even essentially **countable**.

Classifiability by (Polish) abelian groups

The natural action $\ell^1 \curvearrowright \mathbb{R}^{\mathbb{N}}$ is not even essentially **countable**.
$\ell^1 =$ absolutely summable sequences in $\mathbb{R}^{\mathbb{N}}$.
Maybe essential countability is the only obstruction?
Hjorth's conjecture:

$$\begin{array}{c}\text{class}^{\text{ble}}\text{ by abelian}\\ + \\ \text{essentially countable}\end{array} \implies \text{essentially hyperfinite}$$

Equivalently:

$$\begin{array}{c}\text{class}^{\text{ble}}\text{ by abelian}\\ + \\ \text{class}^{\text{ble}}\text{ by countable}\end{array} \implies \text{class}^{\text{ble}}\text{ by countable abelian}$$

Evidence for Hjorth's conjecture

Hjorth's conjecture:

$$\begin{array}{c}\text{class}^{\text{ble}}\text{ by abelian}\\ + \\ \text{essentially countable}\end{array} \implies \text{essentially hyperfinite}$$

Cotton: true if "abelian" is replaced by "locally compact abelian".
Shaun Allison: true if "abelian" is replaced by "$\mathbb{R}^{\mathbb{N}}$".

Hjorth's conjecture refuted!

Recently, Shaun Allison disproved Hjorth's conjecture:

Theorem (Shaun Allison)

Every treeable CBER is classifiable by a Polish abelian group.

In particular, free part of $F_2 \curvearrowright 2^{F_2}$ is a counterexample to Hjorth.
Actually, no restriction needed on the CBER:

Theorem (Frisch-信)

Every CBER is classifiable by a Polish abelian group.

In other words, we simply have

$$\text{class}^{\text{ble}} \text{ by countable} \implies \text{class}^{\text{ble}} \text{ by abelian}$$

Proof idea:

Hardest CBER: Schreier graph of $F_2 \curvearrowright 2^{F_2}$.
Let G be a 4-regular Borel graph.
Want a reduction:

$$\begin{array}{c}\text{connected component relation on } G \\ \downarrow \\ \text{orbit eq rel of a Polish abelian group}\end{array}$$

Basic idea (no DST yet)

Notation: $\mathbb{F}_2[X]$ = free $\mathbb{F}_2$-vector space generated by X.
G: graph with vertices V_G and edges G
Define a cover $\widetilde{G}$ of G:

- Vertices: $\mathbb{F}_2[G] \times V_G$.
- Edges: Neighbors of (a, v)? For every w adjacent to v, $(a + \overline{vw}, w)$ is adjacent to (a, v).

Basic idea cont'd (no DST yet)

Define cover $\widetilde{G}$ of G: vertex set $\mathbb{F}_2[G] \times V_G$, with (a, v) adjacent to $(a + \overline{vw}, w)$.
Consider the natural action $\mathbb{F}_2[G] \curvearrowright \widetilde{G}$ (change first coordinate).
Descend to action $\mathbb{F}_2[G] \curvearrowright \mathrm{Comp}(\widetilde{G})$, the set of components of $\widetilde{G}$.
For vertices v, w in G:

$$v \text{ and } w \text{ are in same component}$$
$$\Updownarrow$$
$$\text{component of } (0, v) \text{ and component of } (0, w) \text{ are in same orbit}$$

i.e. "$v \mapsto$ component of $(0, v)$" is the desired reduction.

DST has issues with this

Let G be a 4-regular Borel graph.
Define cover $\widetilde{G}$ of G: vertex set $\mathbb{F}_2[G] \times V_G$, with (a, v) adjacent to $(a + \overline{vw}, w)$.
We have a reduction from connectedness rel of G to $\mathbb{F}_2[G] \curvearrowright \mathrm{Comp}(\widetilde{G})$!
The issues...

- **Problem 1**: $\mathbb{F}_2[G]$ isn't a Polish group.
  The "free object generated by a Polish space" is usually not Polish, e.g. free group, free abelian group, free vector space...
  Even if we overcome this...
- **Problem 2**: $\mathrm{Comp}(\widetilde{G})$ isn't always standard Borel.
  It's only standard if the connectedness rel on $\widetilde{G}$ is smooth.

The actual plan

X: standard Borel space.
We want a Polish version of "$\mathbb{F}_2[X]$".
Idea:

1. Put a metric on X.
2. This induces a metric on $\mathbb{F}_2[X]$ (the **earthmover distance**).
3. Its completion $\overline{\mathbb{F}_2[X]}$ is a Polish group, use this.

Now suppose we have a Borel graph G, and do the above for $X = G$.
Define the cover $\widetilde{G}$ using vertex set $\overline{\mathbb{F}_2[G]} \times V_G$, with (a, v) adjacent to $(a + \overline{vw}, w)$.
For $\mathrm{Comp}(\widetilde{G})$ to be standard Borel, we need the connectedness rel on $\widetilde{G}$ to be **smooth**.
The smoothness depends on the metric we choose on X!

The earthmover distance on $\mathbb{F}_2[X]$

X: metric space with $d \le 1$.
Define **earthmover norm** on $\mathbb{F}_2[X]$
Intuitively:

Start with a finite set of garbage you need to clean up.
There's a garbage can at distance 1 from everyone.
Two pieces of garbage at the same point will cancel out (like in real life).
What's the smallest amount of garbage movement required?

Fix $F \in \mathbb{F}_2[X]$.
Let $(a_0, b_0), \dots, (a_k, b_k)$ be a (partial) matching of F.

$$\|F\| \le d(a_0, b_0) + \cdots + d(a_k, b_k) + \text{number of unmatched points}$$

Choosing a metric on G

$\widetilde{G}$: cover of G with vertex set $\overline{\mathbb{F}_2[G]} \times V_G$.
Fix vertex v in G.
Every walk $e_0 e_1 \cdots e_k$ starting at v gives a walk in $\widetilde{G}$ starting at $(0, v)$:

$$
\begin{gathered}
(0, v) \\
(e_0, \text{endpoint of } e_0) \\
(e_0 + e_1, \text{endpoint of } e_0 e_1) \\
(e_0 + e_1 + e_2, \text{endpoint of } e_0 e_1 e_2) \\
\vdots \\
(e_0 + e_1 + e_2 + \cdots + e_k, \text{endpoint of } e_0 e_1 \cdots e_k)
\end{gathered}
$$

The component of $(0, v)$ consists of everything of this form.
Choose a metric on G so that the component of $(0, v)$ is **discrete** (implying smooth).

Treeable case

$\widetilde{G}$: cover of G with vertex set $\overline{\mathbb{F}_2[G]} \times V_G$.
Component of $(0, v)$: pairs of the form $(vw\text{-walk}, w)$.
Assume G is **acyclic**.
Component of $(0, v)$: pairs of the form $(vw$-**path**$, w)$.
Choose metric on G so that every path has norm $\gg 0$.
Note: the **discrete metric (with distance 1)** gives norm 5 to every set of size 10.
Fix a Borel map $G \to \mathbb{N}$ which is injective on 10-balls.
Pull back the metric from $\mathbb{N}$ to G.
Then a 10-path consists of 10 different colors, so it has norm 5.

Treeable case (cont'd)

$\widetilde{G}$: cover of G with vertex set $\overline{\mathbb{F}_2[G]} \times V_G$.
G is acyclic, component of $(0, v)$ has pairs of form $(vw$-**path**$, w)$.
At every scale n: For every n, make a Borel map $c_n : G \to \mathbb{N}$ which is injective on n-balls.
Pulling back discrete metric via c_n makes n-paths have norm $\frac{n}{2}$.
Take the supremum over n?
Only possible if metrics go to 0.
Instead, at n-th scale, pull back $\frac{\text{discrete}}{n}$.

Beyond treeable

$\widetilde{G}$: cover of G with vertex set $\overline{\mathbb{F}_2[G]} \times V_G$.
Component of $(0, v)$: pairs of the form $(vw\text{-walk}, w)$.
What to do at scale n?
Problems like, very long walks between adjacent vertices.
Before: maps c_n were "homomorphisms to a complete graph".
Solution: take maps c_n to be "homomorphisms to a graph with larger diameter".
In particular, walk of length 10 has colors which are "10 apart".
It must cross every threshold from 0 to 10 an odd number of times, so that's 10 distinct colors, giving norm 5 again.

Beyond CBERs

We've shown

$$\text{class}^{\text{ble}} \text{ by countable} \implies \text{class}^{\text{ble}} \text{ by abelian}$$

We can push this to countable products:

$$\text{class}^{\text{ble}} \text{ by countable product of countable} \implies \text{class}^{\text{ble}} \text{ by abelian}$$

i.e.

$$\text{class}^{\text{ble}} \text{ by } \textbf{TSI non-archimedean} \implies \text{class}^{\text{ble}} \text{ by abelian}$$

TSI $=$ there is a two-sided invariant metric.
Abelian groups are TSI.
Very natural guess:

$$\text{class}^{\text{ble}} \text{ by TSI} \iff \text{class}^{\text{ble}} \text{ by abelian}$$

The E_2-dichotomy

Recall

$$\text{classifiable by a Polish abelian group} \iff \text{classifiable by } \ell^1$$

where $\ell^1 = \{x \in \mathbb{R}^{\mathbb{N}} : x \text{ is absolutely summable}\}$.
In other words,

Every CBER reduces to some orbit equivalence relation of ℓ^1.

Which orbit equivalence relation?
Denote $E_2 =$ orbit equivalence relation of $\ell^1 \curvearrowright \mathbb{R}^{\mathbb{N}}$.

Question

Does every CBER reduce to E_2?

Hjorth conjecture: **no**, only hyperfinite ones reduce to E_2...

Thank you!