Invariant uniformization and reducibility

Caltech logic seminar

Joint work with Alexander Kechris
Axiom of Choice and Uniformization

X, Y sets

\{P_x\}_{x \in X} a family of non-empty sets P_x \subseteq Y

AC There is a choice function F : X \to Y

\text{s.t. } f(x) \in P_x \ \forall x \in X

Q] Definable AC? Is there a Borel choice fcn?

Q] If P is Borel, is there a Borel choice fcn

\text{Uniformization of } P

\text{Given } P, \text{ we let } P_x = \{y : (x, y) \in P\}

x-section of P
1. **Existence of Borel uniformizations**

   In this talk: $X, Y$ always Polish uniformizations $P \subseteq X \times Y$ always Borel.

   **Standard results** If the sections of $P$ are...

   "large section" category $P_x$ is non-meagre for all $x$ measure $P_x$ is $\mu$-positive for all $x$.

   "small section" countable $P_x$ is countable for all $x$.

   Lusin, Novikov, Kuratowski $P_x$ is $K_\sigma$ for all $x$, $\cap$ ctbl union of compact sets...

   Arsenin, Kunugi... then $P$ admits a Borel uniformization.
Invariant uniformization

$E$ Borel equivalence relation on $X$

$P$ $E$-invariant $\quad x \in x' \Rightarrow P_x = P_{x'}$

$f$ s.t. $x \in x' \Rightarrow f(x) = f(x')$

+ “large” or “small” sections

Q: Is there a Borel $E$-invariant uniformization?

Eq. 0: $E$ a countable Borel equivalence relation

\( \Rightarrow \) all equiv. class are countable

\( Y = X, \ P = E \subseteq X \times X \)

An $E$-inv. uniformization $\Rightarrow$ A Borel selector

\( \Rightarrow \) There is an $E$-inv. uni. $\quad \Leftrightarrow \quad E$ is smooth

\( E_0 = P \in F_o \)
$E \otimes (X = Y = 2^\omega, \ E = E_0 = "eventual\ equality")$

$x \ E_0 y \iff \exists n \geq m (x_m = y_m)$

Let $\mu$ be the uniform measure on $2^\omega$

Let $A$ be a set which is $E_0$-inv, comeagre and $\mu$-null.

$(x, y) \in P \iff x \in A + y$

- $P$ has comeagre sections + is $E_0$-inv
- For all $y$, $P_y = \{x : (x, y) \in P\}$ is $\mu$-null

$P$ does not admit an $E_0$-inv, uniform, $\mu$-comeagre sections + $E_0$-inv.

If $f$ were such a uniform, then $f(\mathcal{B})$ is const.

If $f(x) = y$ for $x \in B$, then $B \in P_y$.

Can choose $A \in G_5 \implies P \in G_5$.

got $P$ unmeagre $\mu$-comeagre sections + no inv. unif. $P \in E_0$.
A "global" characterization

Def: $E$ is smooth if there exists a Polish space $\mathcal{B}$ and a map $f: X \to \mathcal{B}$ such that $x \in E_y \iff f(x) = f(y)$

Thm: The following are equivalent:

1. $E$ is smooth

2. For all $P$, if $P$ has non-meagre sections ($E$ is $E$-inv)
   then there exists an $E$-invariant uniformization

3. For all $P$, if $P$ has $\mu$-positive sections
   then there exists an $E$-invariant uniformization

4. For all $P$, if $P$ has countable sections
   then there exists an $E$-invariant uniformization

5. For all $P$, if $P$ has $K_0$ sections
   then there exists an $E$-invariant uniformization
3.1 Complexity of counter examples

Q1 If \( P \) has “large” or “small” sections and has no \( E \)-invariant uniformization, how “complicated” can \( P \) be?

\[ \text{Borel complexity} \]

\[ \begin{align*}
\text{Ex} & \quad \text{There exists } P \text{ such that one of:} \\
\text{category} & \quad P \in Gs & \text{has comeagre sections} \\
\text{measure} & \quad P \in Fr & \text{has conull sections} \\
\text{countable} & \quad P \in F & \text{has ctbl sections}
\end{align*} \]

\[ \text{Is this optimal?} \]

Thm If the sections of \( P \) satisfy one of the following:

\[ \begin{align*}
\text{category} & \quad P \in Fr & \text{is non-meagre} \\
\text{measure} & \quad P \in \Delta^2 & \mu \text{-pos.} \\
\text{K}\alpha & \quad P \in Gs & \text{K}\alpha
\end{align*} \]

then there is an \( E \)-invariant uniformization.

Thm There exists \( E, P \) such that \( P \) has no \( E \)-invariant uniformization, but \( P \in Gs \) and has sections that are both comeagre \& conull.
Proof sketch: Let $X = \mathcal{P}(\omega)^\omega \subseteq 2^\omega$, $E = E_\emptyset \cap X$.

Find $P \subseteq X \times Y$ such that:
- $P \in G$
- $P$ has comeagre, conull sections
- $P^y = \{x : (x,y) \in P^3\}$ is Ramsey-null for all $y \in Y$

Examples of such $P$:

1. $Y = 2^\omega$, $P(A, B) \iff |A \setminus \emptyset| = |A \cap B| = \aleph_0$

2. $Y = \{\text{graphs on } \omega^3\}$, $P(A, G) \iff A$ "witnesses" that $G$ is the random graph

3. $Y = \{\text{strictly increasing functions } f : \omega \to \omega^3\}$, $P(A, f) \iff f(A)$ contains infinitely many even & odd $\#s$
“Local” dichotomies and anti-dichotomies

We characterized those $E$ such that all $P$ with “large” or “small” sections have an $E$-invariant uniformization.

What about characterizing the pairs $(E, P)$ which admit invariant uniformizations?

**Theorem (Miller)** Suppose $P$ is $E$-invariant and has countable sections.

Exactly one of the following holds:

1. There is an $E$-invariant uniformization

2. There is a continuous embedding of the pair $(E \times I_w, E \times I_w)$ into the pair $(E, P)$

$E \times I_w$ equiv. rel. on $2^\omega \times \omega_1$ and $(2^\omega \times 2^\omega)^2$

$(x, \eta) \sim E \times I_w \iff (y, \eta) \in x \times E \circ y$
Embedding $(E_0 \times l_w, E_0 \times l_w)$ into $(E, P)$
4.1 Comment on proof

Miller's proof: - proves more general dichotomy
- proves dichotomy "from scratch" (using idea of "puncture sets")

We give two new proofs:
1. Uses "off-the-shelf" dichotomies $\sim (G_0, H_0)$ [Miller], $\aleph_0$-dimensional $G_0$ [Lecomte]
2. Follows from a new "$\aleph_0$-dimensional $(G_0, H_0)$" dichotomy

Thm. There is an $\aleph_0$-dimensional graph $G_0^w$ on $w^w$, and a graph $H_0^w$ on $w^w$, such that for all $\aleph_0$-dimensional analytic graphs $G$ on $X$ and all analytic equivalence relations $E$ on $X$, exactly one of the following holds:

1. There is a smooth Borel equivalence relation $F \equiv E$ and a countable Borel $F$-local colouring of $G$
2. There is a continuous homomorphism $\chi : X_\alpha \rightarrow X$ of $(G_0^w, H_0^w)$ to $(G, E)$

\begin{align*}
\text{strictly increasing} & \quad X_\alpha = \{x \in w^w : x \leq e \cdot \alpha(\omega) \} \quad \text{of the}
\end{align*}
4.2 Anti-dichotomy results

Q) What about for \( P \) with "large" sections?

Dichotomies give bounds on the "complexity" of problems:

**Thm** The (codes of) pairs \((E, P)\), where \( P \) has countable sections and admits an \( E \)-invariant uniformization, is \( \Pi_1 \).

**Thm** The (codes of) pairs \((E, P)\), where \( P \) has "large" sections and admits an \( E \)-invariant uniformization, is \( \Sigma_2 \)-complete.

**Note** This holds even when \( E, P \) are "simple".

Open problem Is there a dichotomy for the case of \( K_0 \) sections?

\[ E = R \cdot E_0 \]
Invariant ctbl uniformizations

Inv. unif: choose a point from every section in an inv. way

Can we choose a ctbl set of pts from each section in an inv. way?

\[ f : X \to Y^m \]

\[ x \in E \implies \{ f(x)_n \} = \{ f(x')_n \} \]

By Lusin-Novikov if \( P \) has ctbl sections this is always possible.

Lemma: If \( E \) fails inv. ctbl unif. to \( K_\sigma \) (resp. non-meagre, \( \mu \)-pos.) sections and \( E \subseteq E' \) then so does \( E' \) ctbl inv. unif.

when \( E \) is red. to ctbl, \( E \subseteq E' \)
then \( E \) admits ctbl inv. unif.
(Q) If $E$ is not red to $\mathsf{ctbl}$, does $E$ fail inv. $\mathsf{ctbl. uni.}$ when the sections are "large" or $K_0$?

\text{for } E_1, E_2 \text{ fail inv. } \mathsf{ctbl uni.} \text{ when the sections are } K_0

\begin{align*}
E_1 \text{ on } (\mathbb{2})^\omega &\wedge E_1 y \iff \exists n \forall m (x_n = y_m) \\
E_2 \text{ on } \mathbb{2}^\omega &\wedge E_2 y \iff \exists n \frac{1}{2^n} < \infty
\end{align*}

$E_0$ is $E_1$-inv. & has co-$\mathsf{ctbl}$ sections
but has no $E_1$-inv. $\mathsf{ctbl uni.}$

\text{Proof } E_{ctbl} \text{ on } (\mathbb{2})^\omega \wedge E_{ctbl} y \iff \exists x, 3 = 3y, 3

fails inv. $\mathsf{ctbl}$ uni. when the sections are "large"

(Q) Does $E_{ctbl}$ fail inv. $\mathsf{ctbl uni.}$ for $K_0$-sections?
Thank you