Invariant uniformization and reducibility

Caltech logic seminar

Joint work with Alexander Kechris

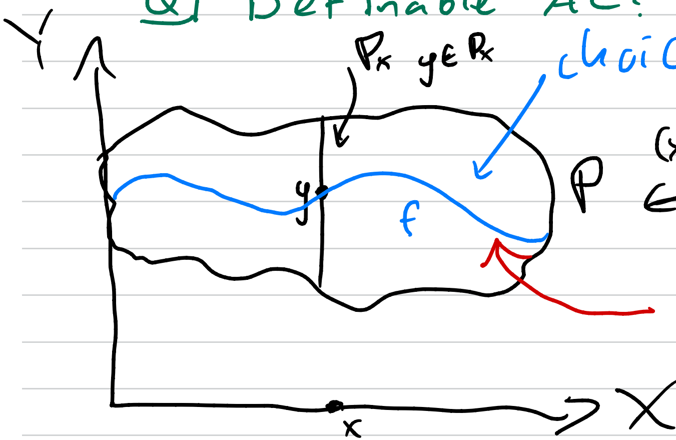
① Axiom of Choice and Uniformization

X, Y ~~sets~~ Polish spaces

$\{P_x\}_{x \in X}$ a Borel family of non-empty sets $P_x \subseteq Y$

<u>AC</u> There is a choice function $f : X \to Y$
s.t. $f(x) \in P_x \quad \forall x \in X$

<u>Q)</u> Definable AC? Is there a Borel choice fcn?



choice fcn

$(x,y) \in P \iff y \in P_x$

P is Borel

<u>Q</u> If P is Borel, is there a Borel choice fcn

uniformization of P

Given P, we let
$P_x = \{y : (x,y) \in P\}$
x-section of P

(1.1) Existence of Borel uniformizations

all sections are non-$\emptyset$

In this talk: X, Y always Polish
$P \subseteq X \times Y$ always Borel

uniformizations are Borel

Standard results If the sections of P are...

μ is a Borel prob. meas. on Y

"large section" <u>category</u> P_x is non-meagre for all x

<u>measure</u> P_x is μ-positive for all x

"small section" <u>countable</u> P_x is countable for all x

Lusin–Novikov

<u>K_σ</u> P_x is K_σ for all x

ctbl union of compact sets

Arsenin–Kunugui

... then P admits a Borel uniformization

② Invariant uniformization

E Borel equivalence relation on X

P E-invariant $x E x' \Rightarrow P_x = P_{x'}$

+ "large" or "small" sections

f s.t. $x E x' \Rightarrow f(x) = f(x')$

Q| Is there a Borel E-invariant uniformization?

Eg ① E a countable Borel equivalence relation

all equiv. class are countable

$Y = X$, $P = E \subseteq X \times X$

An E-inv. uniformization $\Leftrightarrow$ A Borel selector

$\leadsto$ There is an E-inv. unif. $\Leftrightarrow$ E is smooth

$E_0 = P \in F_\sigma$

Eg ② $X = Y = 2^\omega$, $E = E_0$ = "eventual equality"

$$x \, E_0 \, y \Leftrightarrow \exists n \, \forall m \geq n \, (x_m = y_m)$$

Let μ be the uniform measure on 2^ω

Let A be a set which is E_0-inv, comeagre and μ-null

$(x,y) \in P \Leftrightarrow x \in A + y$ coordinate-wise addition mod 2

- P has comeagre sections + is E_0-inv
- For all y, $P^y = \{x : (x,y) \in P\}$ is μ-null

P does not admit an E_0-inv. unif.
If f were such a unif, then $f \upharpoonright B$ is const. (B μ-conull & E_0-inv)

$\leadsto$ if $f(x) = y$ for $x \in B$, then $B \subseteq P^y$ ↯

can choose $A \in G_\delta \rightarrow P \in G_\delta$
get P w/ meagre μ-conull sections & no inv. unif. $P \in F_\sigma$

③ A "global" characterization

Polish space

Def^n E is Smooth if $\exists$ Borel map $f: X \to Z$ s.t. $x E y \Leftrightarrow f(x) = f(y)$

Thm The following are equivalent:

① E is smooth

② For all P, if P has non-meagre sections (& is E-inv) then there exists an E-invariant uniformization

③ For all P, if P has μ-positive sections then there exists an E-invariant uniformization

some Borel prob. meas. μ on Y

④ For all P, if P has countable sections then there exists an E-invariant uniformization

⑤ For all P, if P has K_σ sections then there exists an E-invariant uniformization

3.1 Complexity of counter examples

Q] If P has "large" or "small" sections and has no E-invariant uniformization, how "complicated" can P be?
Borel complexity

Eg There exists P such that one of:

yes → category $P \in G_\delta$ & has comeagre sections
No → measure $P \in F_\sigma$ & has conull sections
yes → countable $P \in F$ & has ctbl sections

holds, but there is no E-invariant uniformization

Is this optimal?

Thm If the sections of P satisfy one of the following:

category $P_x \in F_\sigma$ & is non-meagre
measure $P_x \in \Delta^0_2$ & μ-pos.
K_σ $P_x \in G_\delta$ & K_σ

then there is an E-invariant uniformization

Thm There exists E, P such that P has no E-invariant uniformization, but $P \in G_\delta$ and has sections that are both comeagre & conull inf. subsets of ω

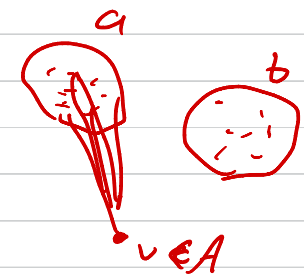
Proof sketch Let $X = [\omega]^\omega \subseteq 2^\omega$, $E = E_0 \upharpoonright X$

If f were a unif. $f \upharpoonright [A]^\omega$ would be const.

↑ inf. subsets of $A \in [\omega]^\omega$

Find $P \subseteq X \times Y$ such that:

- $P \in G_\delta$
- P has comeagre, conull sections
- $P^y = \{x : (x,y) \in P\}$ is Ramsey-null for all $y \in Y$



Examples of such P

① $Y = 2^\omega$ $\quad P(A,B) \Leftrightarrow |A \setminus B| = |A \cap B| = \aleph_0$

② $Y = \{\text{graphs on } \omega\}$, $\quad P(A,G) \Leftrightarrow A$ "witnesses" that G is the random graph

③ $Y = \{\text{strictly increasing fcns } f : \omega \to \omega\}$

$P(A,f) \Leftrightarrow f(A)$ contains ∞-ly many even & odd #s

④ "Local" dichotomies and anti-dichotomies

Q1 We characterized those E such that all P with "large" or "small" sections have an E-invariant uniformization.

What about characterizing the pairs (E, P) which admit invariant uniformizations?

Thm (Miller) Suppose P is E-invariant and has countable sections. Exactly one of the following holds:

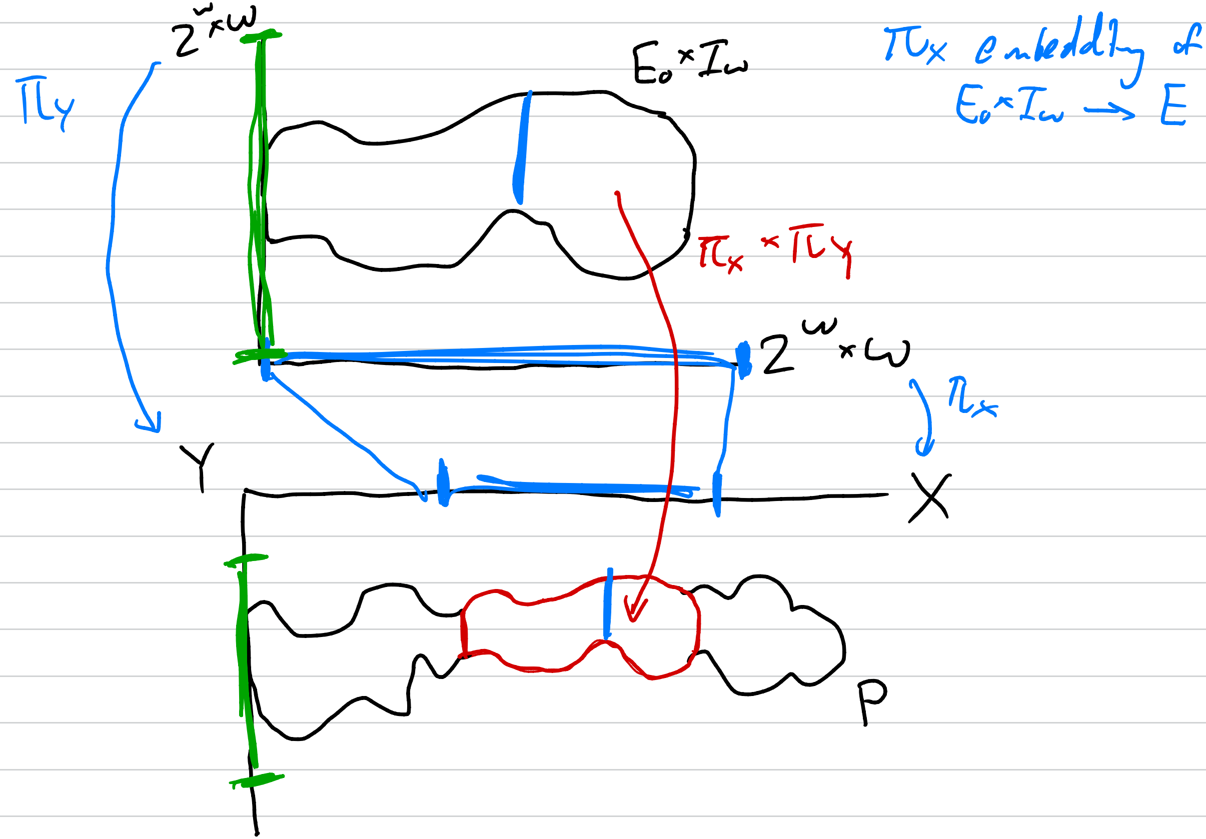
① There is an E-invariant uniformization

② There is a continuous "embedding" of the pair $(\underset{E}{E_0 \times I_\omega}, \underset{P}{E_0 \times I_\omega})$ into the pair $\underset{(2^\omega \times \omega)^2}{(E, P)}$

$E_0 \times I_\omega$ equiv. rel. on $2^\omega \times \omega$

$(x, n)\ E_0 \times I_\omega\ (y, m) \iff x\ E_0\ y$

Embedding $(E_0 \times I_\omega, E_0 \times I_\omega)$ into (E, P)



4.1 Comment on proof

Miller's proof: - proves more general dichotomy
- proves dichotomy "from scratch" (using idea of "puncture sets")

We give two new proofs
1. Uses "off-the-shelf" dichotomies → (G_0, H_0) [Miller], $\aleph_0$-dimensional G_0 [Lecomte]
2. Follows from a new "$\aleph_0$-dimensional (G_0, H_0)" dichotomy

Thm There is an $\aleph_0$-dimensional graph G_0^ω on ω^ω, and a graph H_0^ω on ω^ω, such that for all $\aleph_0$-dimensional analytic graphs G on X and all analytic equivalence relations E on X, exactly one of the following holds:

1. There is a smooth Borel equivalence relation $F \supseteq E$ and a countable Borel F-local colouring of G — $c \restriction B$ is a colouring of $G \restriction B$ $\forall F$-classes B
2. There is a continuous homomorphism $\varphi : X_\alpha \to X$ of (G_0^ω, H_0^ω) to (G, E)

$\alpha : \omega \to \omega$ strictly increasing

$X_\alpha = \{x \in \omega^\omega : x \restriction n \in \alpha(n)^n \text{ } \infty\text{-ly of them}\}$

4.2 Anti-dichotomy results

Q) what about for P with "large" sections?

Dichotomies give bounds on the "complexity" of problems:

either non-meagre or μ-pos

Thm The (codes of) pairs (E,P), where P has countable sections and admits an E-invariant uniformization, is $\mathbf{\Pi}^1_1$

Thm The (codes of) pairs (E,P), where P has "large" sections and admits an E-invariant uniformization, is $\mathbf{\Sigma}^1_2$-complete

Note This holds even when E, P are "simple" $\rightarrow$ E hyperfinite
P comeagre sections & is G_δ
or P has countbl sections & is F_σ

Open problem Is there a dichotomy for the case of K_σ sections?

$$E = \mathbb{R} \cdot E_0$$

Invariant ctbl uniformizations

Inv. unif: choose a point from every section in an inv. way

Can we choose a ctbl set of pts from each section in an inv. way?

$$f: X \longrightarrow Y^{\mathbb{N}}$$

$$x E x' \Rightarrow \{f(x)_n\} = \{f(x')_n\}$$

By Lusin-Novikov if P has ctbl sections this is always possible

Lemma If E fails inv. ctbl unif. for K_σ (resp. non-meagre, μ-pos.) sections and $E \leq_B E'$ then so does E' ctbl Borel equiv. rel.

when E is red. to ctbl, $E \leq_B E'$ then E admits ctbl inv. unif.

Q) If E is not red. to ctbl, does E fail inv. ctbl. unif. when the sections are "large" or K_σ?

Prop^n E_1, E_2 fail inv. ctbl unif. when the sections are K_σ

E_1 on $(2^\omega)^\omega$ $\quad x E_1 y \Leftrightarrow \exists n \forall m \geq n (x_m = y_m)$

E_2 on 2^ω $\quad x E_2 y \Leftrightarrow \sum_{n : x_n \neq y_n} \frac{1}{n+1} < \infty$

Eg P is E_1-inv. & has co-ctbl sections but has no E_1-inv. ctbl unif.

Prop^n E_{ctbl} on $(2^\omega)^\omega$ $\quad x\, E_{ctbl}\, y \Leftrightarrow \{x_n\} = \{y_n\}$

fails inv. ctbl. unif. when the sections are "large"

Q) Does E_{ctbl} fail inv. ctbl unif. for K_σ-sections?

Thank you