Translational tilings of the plane by a polygonal set

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- joint work in progress with de Dios Pont, Greenfeld and Madrid

$F + a = \{f + a : f \in F\}$

Theorem (Newman)

Let $F \subseteq \mathbb{Z}$ be finite and $A \subseteq \mathbb{Z}$ be such that

$$F \oplus A = \bigsqcup_{a \in A} (F + a) = \mathbb{Z}.$$

Then A is periodic. $\exists k \in \mathbb{N} \setminus \{0\}$ s.t. $A + k = A$

Conjecture (Periodic tiling conjecture (PTC))

$d > 1$ *Let $F \subseteq \mathbb{Z}^d$ be finite and $A \subseteq \mathbb{Z}^d$ be such that $\mathbb{Z}^d = F \oplus A$. Then there is $B \subseteq \mathbb{Z}^d$ that is periodic such that $\mathbb{Z}^d = F \oplus B$.*

- false for large d (Greenfeld and Tao),
- correct for $d = 2$ (Bhattacharya)

Tilings in $\mathbb{R}^2$

$\Omega \;\equiv$ *bounded measurable set* of positive measure $\equiv$ **tile**

$T \subseteq \mathbb{R}^2 \;\equiv$ set of translates

Definition

We say that Ω *tiles* $\mathbb{R}^2$ by T, $\underbrace{\mathbb{R}^2 = \Omega \oplus T}$, if

(a) for every $x \in \mathbb{R}^2$ there is $t \in T$ such that $x \in \Omega + t$,

(b) for every $t \neq s \in T$ we have that $(\Omega + t) \cap (\Omega + s)$ is null.

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Remark:

- only translations, no rotations,
- Ω is a single tile, it might be disconnected,
- there is $r_\Omega > 0$, that depends only on Ω, such that every such T is r_Ω-separated. $t \neq s \in T \quad \|t-s\|_2 > r_\Omega$

Tilings in $\mathbb{R}^2$

$\exists g, h \in \Gamma(T) \setminus \{(0,0)\}$

s.t. $g \notin \mathbb{R}h$

Given $X \subseteq \mathbb{R}^2$, denote as

$$\Gamma(X) = \{g \in \mathbb{R}^2 : X + g = X\}$$

the set of translational symmetries of X.

Definition

We say that $T \subseteq \mathbb{R}^2$ is periodic if $\Gamma(T)$ contains a lattice.

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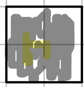
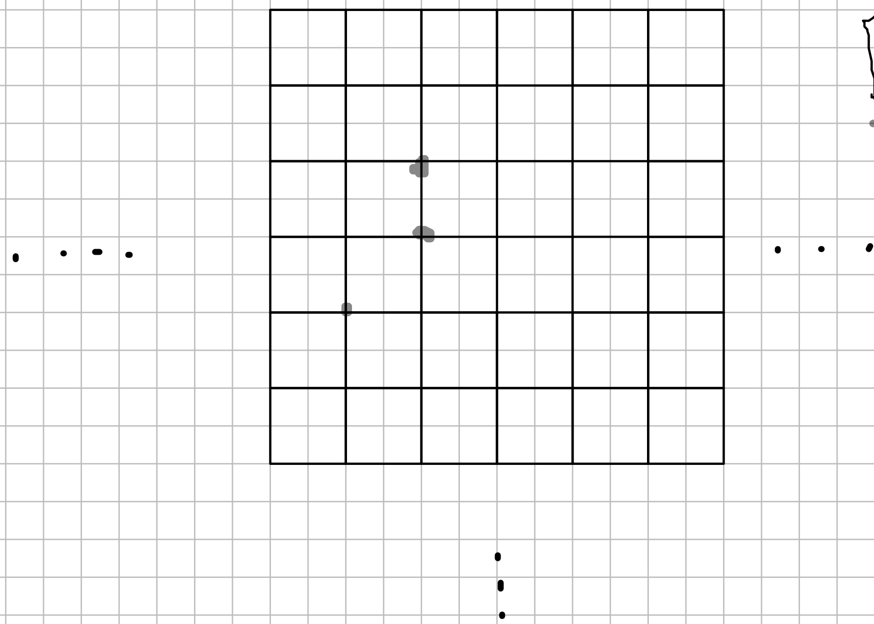
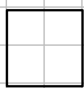
Conjecture (PTC in $\mathbb{R}^2$)

Suppose that Ω tiles $\mathbb{R}^2$ by T. Then there is $S \subseteq \mathbb{R}^2$ that is periodic such that $\mathbb{R}^2 = \Omega \oplus S$.

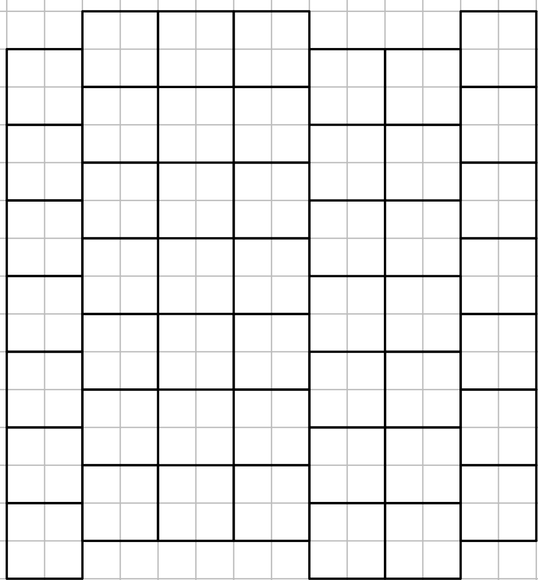
(a) $\Omega =$  , $T = (2\mathbb{Z})^2$



$\Gamma(T) = (2\mathbb{Z})^2$

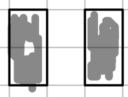
(b) $\Omega =$  , $T = \{(2n, 2m + \alpha_n) : n, m \in \mathbb{Z}\}$

$\alpha_n \in \mathbb{R}$

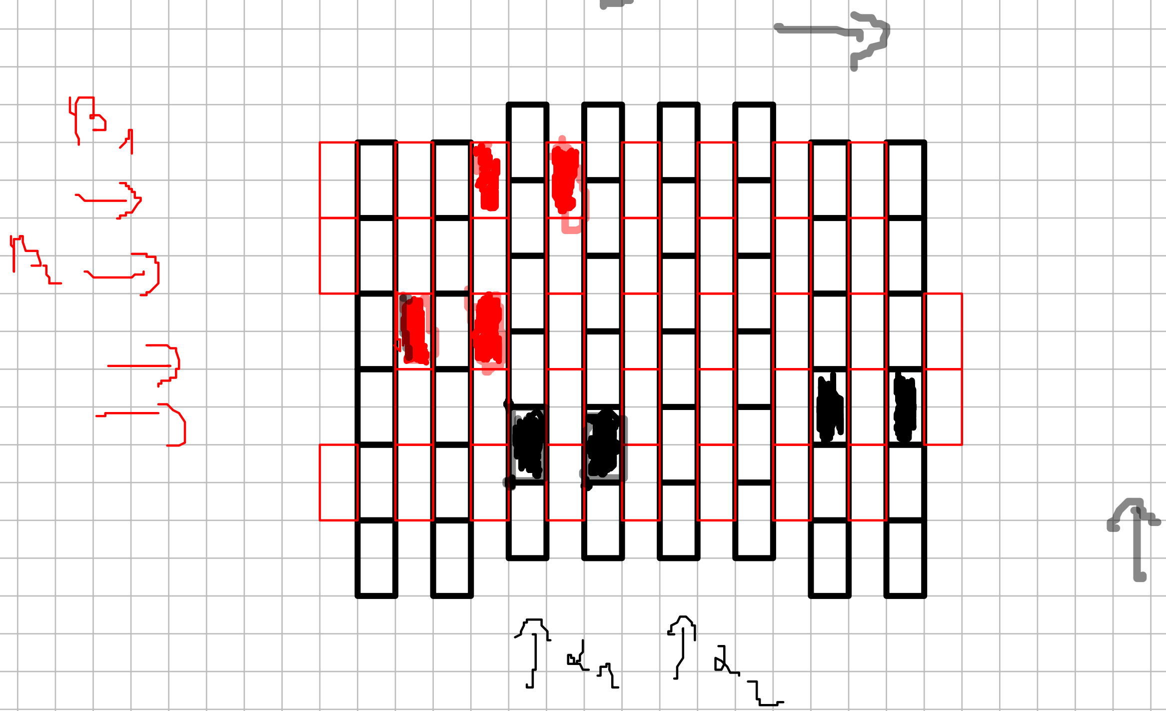


$\Gamma(T) = \mathbb{Z}(0,2)$

$\uparrow \alpha_n = \{0,1\}$

(c) $\Omega =$  disconnected

$T = T_1 \sqcup T_2$



$\Gamma(T) = \{(0,0)\}$

$\Gamma(T_1) = \mathbb{Z}(0,2)$

$\Gamma(T_2) = \mathbb{Z}(4,0)$

Weak periodicity

Definition

We say that $T \subseteq \mathbb{R}^2$ is weakly periodic if we can write

$$T = T_1 \sqcup \cdots \sqcup T_m$$

such that $\Gamma(T_i)$ is non-trivial for every $1 \leq i \leq m$.

Remark:

- $T = T_1 \sqcup T_2$ in example (c) is weakly periodic and $\Gamma(T)$ is trivial,
- weak periodicity is enough for PTC in $\mathbb{Z}^2$.

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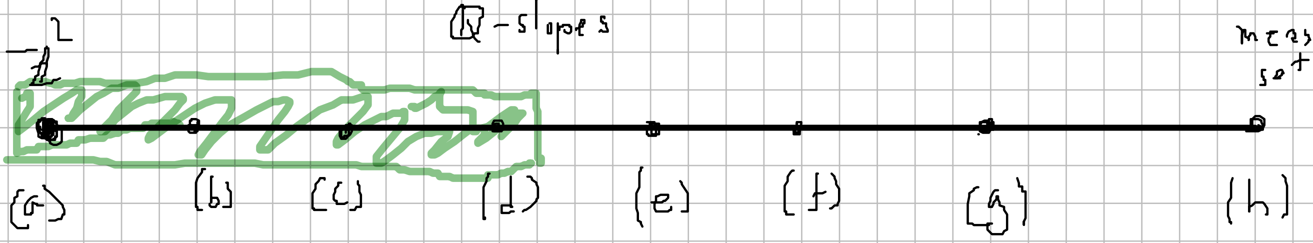
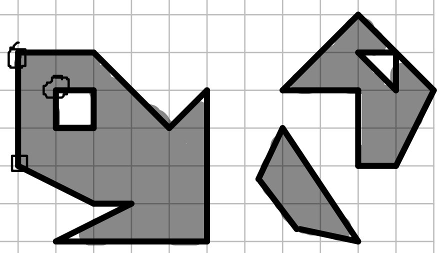
- weak periodicity is enough for PTC in $\mathbb{Z}^2$.

Theorem (Greenfeld–Tao)

Let $F \subseteq \mathbb{Z}^2$ be finite and $A \subseteq \mathbb{Z}^2$ be such that $\mathbb{Z}^2 = F \oplus A$. Then A is weakly periodic.

Complexity of Ω



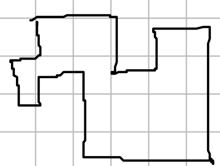


Polygonal set

(a) Ω is fin union $\boxed{}$ + T lies in a lattice $\Big\} = \mathbb{Z}^2$ tilings

(b) polygonal set with $\mathbb{Q}$ vertices

(c) polygonal set with axis parallel edges



(d) polygonal set with $\mathbb{Q}$ slopes

lie in a coset of $G \subseteq \mathbb{R}^2$ that contains $\mathbb{Q}^2 \setminus \{(0,0)\}$

(e) polygonal set

(f) C^1-boundary

(g) closure of its interior

(h) measurable set

Our results

Theorem

Let Ω be a polygonal set with rational slopes and $\mathbb{R}^2 = \Omega \oplus T$ be topologically minimal. Then T is weakly periodic.

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$\Gamma(\xi)$ contains lattice

Remarks:

- the result holds for tilings of a periodic set $E = \Omega \oplus T$,

Our results

weak top = pointwise conv
on bounded sets

Theorem

Let Ω be a polygonal set with rational slopes and $\mathbb{R}^2 = \Omega \oplus T$ be topologically minimal. Then T is weakly periodic.

Remarks:

- the result holds for tilings of a periodic set $E = \Omega \oplus T$,
- the set of all tilings by Ω

$$\mathbf{T}_\Omega = \{ T \subseteq \mathbb{R}^2 : \mathbb{R}^2 = \Omega \oplus T \}$$

  endowed with the weak topology and the action of $\mathbb{R}^2$ by translations, compact

Our results

Using Zorn's lemma $\Rightarrow$ top. minimal always exists.

Theorem

Let Ω be a polygonal set with rational slopes and $\mathbb{R}^2 = \Omega \oplus T$ be topologically minimal. Then T is weakly periodic.

$= T_1 \sqcup \ldots \sqcup T_m \quad \Lambda(T_i)$ nontrivial

Remarks:

- the result holds for tilings of a periodic set $E = \Omega \oplus T$,
- the set of all tilings by Ω

$$\mathbf{T}_\Omega = \{ T \subseteq \mathbb{R}^2 : \mathbb{R}^2 = \Omega \oplus T \}$$

  endowed with the weak topology and the action of $\mathbb{R}^2$ by translations,
- T is **minimal** if there is a sequence $(h_m)_m \subseteq \mathbb{R}^2$ such that $(S + h_m) \to T$ whenever $(T + g_n) \to S$ for some sequence $(g_n)_n \subseteq \mathbb{R}^2$.

  $\forall g_n \quad \exists h_m$

Our results

$$\bigcap_{i=1}^{m} \Gamma(S_i) \text{ is a lattice}$$

Theorem (weak PTC)

Let Ω be a polygonal set with rational slopes and $\mathbb{R}^2 = \Omega \oplus T$. Then there is $S = S_1 \sqcup \cdots \sqcup S_m$ such that $\mathbb{R}^2 = \Omega \oplus S$ and S_i is periodic for every $1 \le i \le m$.

Moreover, if T is locally finite, then there is such S that is periodic.

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Let Ω be a polygonal set with rational slopes and $\mathbb{R}^2 = \Omega \oplus T$. Then there is $S = S_1 \sqcup \cdots \sqcup S_m$ such that $\mathbb{R}^2 = \Omega \oplus S$ and S_i is periodic for every $1 \leq i \leq m$.

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Remarks:

- the result holds for tilings of a periodic set $E = \Omega \oplus T$ and is optimal in this setting,

Our results

Theorem

Let Ω be a polygonal set with rational slopes and $\mathbb{R}^2 = \Omega \oplus T$. Then there is $S = S_1 \sqcup \cdots \sqcup S_m$ such that $\mathbb{R}^2 = \Omega \oplus S$ and S_i is periodic for every $1 \leq i \leq m$.

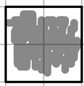
Moreover, if T is locally finite, then there is such S that is periodic.

Remarks:

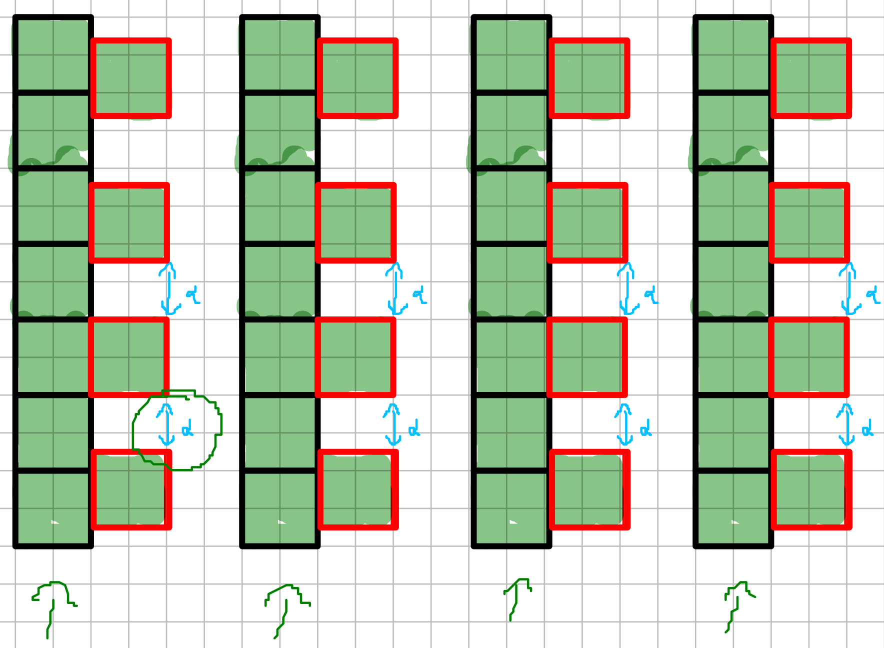
- the result holds for tilings of a periodic set $E = \Omega \oplus T$ and is optimal in this setting,
- T is **locally finite** if for every $R > 0$ we have that

$$|\{\mathcal{B}_R \cap (T - t) : t \in T\}| < \infty.$$

Optimality of the theorem for tilings of a periodic set

$\Omega =$  , $T = \underline{T_1 \sqcup T_2}$ $\square$

$E =$



$$\Gamma(E) \supseteq \mathbb{Z}(6,0),\ \mathbb{Z}(0,2+\alpha)$$

$$\Gamma(T) = \mathbb{Z}(6,0)$$

$$\Gamma(T_1) = 6\mathbb{Z} \times 2\mathbb{Z}$$

$$\Gamma(T_2) = 6\mathbb{Z} \times (2+\alpha)\mathbb{Z}$$

$\nexists\, S \subseteq \mathbb{R}^2$ periodic s.t. $\Omega \oplus S = E$

High-level overview of proofs

$\Omega \oplus T = \mathbb{R}^2$

(1) T minimal $\Rightarrow$ weak periodicity
(2) weak PTC holds

} polygonal set
$\mathbb{Q}$-slopes

(a) Structure of tilings in $\mathbb{Z}^2$ (Greenfeld and Tao)

(b) Rational approximations of $\mathbb{R}^2 = \Omega \oplus T$

} T locally finite

(c) Earthquake decomposition of $\mathbb{R}^2 = \Omega \oplus T$

Discrete tilings

(a) structure of tilings in $\mathbb{Z}^2$ (Greenfeld and Tao)

- every tiling $\mathbb{Z}^2 = F \oplus A$ is weakly periodic, *dilation lemma*,
- in fact, we use *all* the deep structure results from that paper,
- tilings of $\mathbb{Z}^2$ $\equiv$ lattice tilings of $\mathbb{R}^2$ by a tile that is a finite union of unit squares,
- new structure result about periodicity of earthquake decompositions.

Approximations

(b) rational approximations of $\mathbb{R}^2 = \Omega \oplus T$

$$\mathbb{R}^2 = \Omega \oplus T \underset{\varphi}{\rightsquigarrow} \mathbb{R}^2 = \Omega^\varphi \oplus T^\varphi$$

Approximations

(b) rational approximations of $\mathbb{R}^2 = \Omega \oplus T$

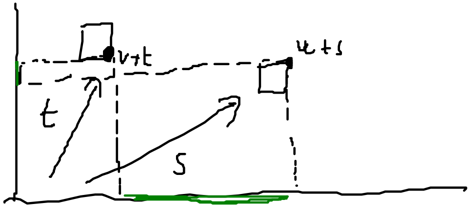
$$\mathbb{R}^2 = \Omega \oplus T \ \rightsquigarrow \ \mathbb{R}^2 = \Omega^\varphi \oplus T^\varphi$$

- let $K_{(\Omega,T)}$ be the $\mathbb{Q}$-linear space generated by

$$\{p_i((w+s)-(v+t)) : v, w \in V(\Omega), s, t \in T, i \in \{1,2\}\},$$

$\Omega = \square$

$s, t \in T$



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- $\varphi : K_{(\Omega,T)} \to \mathbb{Q}$ be a $\mathbb{Q}$-linear map,
- to show that $\mathbb{R}^2 = \Omega^\varphi \oplus T^\varphi$, we need to assume that Ω has rational slopes,
- (a) and (b) enough to handle locally finite T.

Earthquakes

(Kenyon) PTC holds for topological disks.

(c) earthquake decomposition of $\mathbb{R}^2 = \underline{\Omega \oplus T}$

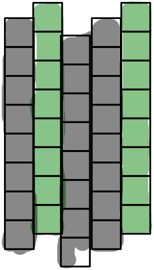
$T \leq \mathbb{R}^2$ some 1-dim. lin subsp.

Definition

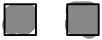
We say that $\mathbb{R}^2 = \Omega \oplus T$ admits a δ-*earthquake* if $T = S_0 \sqcup S_1$ such that $S_0, S_1 \neq \emptyset$ and $\Omega \oplus S_i$ is a union of cosets of δ for $i \in \{0, 1\}$.

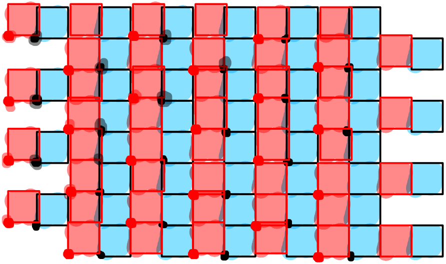
$\Omega =$ 

$T = \mathbb{R}(0,1)$



$S_0 \sqcup S_1$

$\Omega =$ 

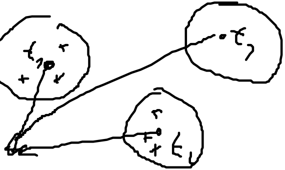


$S_0 \sqcup S_1$

Baby earthquake dichotomy (Kenyon)

$\mathbb{R}^2 = \Omega \oplus T$, where Ω is connected and T is not locally finite.

That is, there is $(t_n)_n$ and $R > 0$ such that



$$(\mathcal{B}_R \cap (T - t_n))_n$$

are pairwise distinct.

Claim

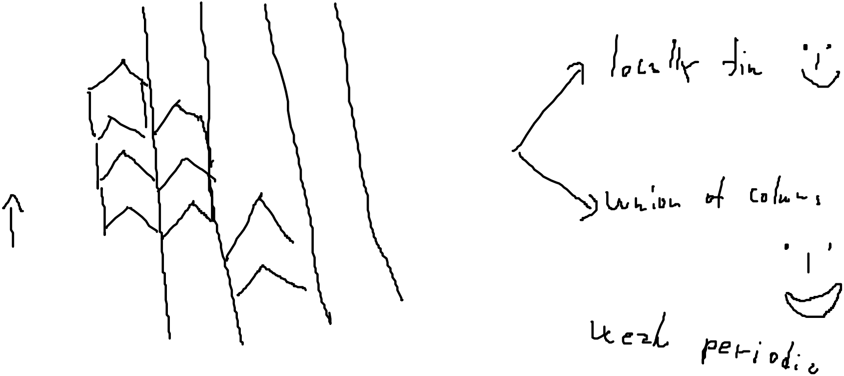
Let $(T - t_n) \to S$. Then $\mathbb{R}^2 = \Omega \oplus S$ contains a δ-earthquake.

$T_n = T - t_n \quad \rightsquigarrow \quad \Omega \oplus \overline{T_n} = \mathbb{R}^2$

Baby earthquake dichotomy (Kenyon)

Theorem

Let Ω be connected, T be minimal and $\mathbb{R}^2 = \Omega \oplus T$. Then either T is locally finite or it is a union of columns.



Earthquake dichotomy

Theorem

Let Ω be a polygonal set with rational slopes, T be minimal and $\mathbb{R}^2 = \Omega \oplus T$. Then either T is locally finite or there is $T = T_0 \sqcup T_1$ that is a δ-earthquake and $\underline{\Omega \oplus T_i}$ is periodic for $i \in \{0, 1\}$.

Earthquake dichotomy

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Remark:

- new even for $\mathbb{Z}^2$,

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Remark:

- new even for $\mathbb{Z}^2$,
- after iterating one get that $T = T_0 \sqcup \cdots \sqcup T_m$ where for every $0 \leq i \leq m$ we have that $\Omega \oplus T_i$ is periodic and T_i is either locally finite or $\Omega \oplus T_i$ is a union of columns.

kerl PTC $\quad S = S_0 \sqcup \ldots \sqcup S_m$

Thank you!