Pi-1-1 maximal almost disjoint families and Laver measurability

Caltech seminar, January 22, 2025
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Definability, mad families, and regularity properties

"Mad" families of subsets of ω, i.e. *maximal almost disjoint* families, are combinatorial objects in set theory that have been investigated very intensively for their definability properties.

Definition

- $[\omega]^\omega$ is the set of all **infinite** subsets of ω.
- Sets $A, B \in [\omega]^\omega$ are almost disjoint if $A \cap B$ is finite.
- An almost disjoint family is a collection $\mathcal{A} \subseteq [\omega]^\omega$ which consits of pairwise almost disjoint sets.
- A **mad** family is an *infinite* **m**aximal (under $\subseteq$) **a**lmost **d**isjoint family.

Some theorems

1. Mathias (1969): There are no analytic mad families.
2. T. (2014): There are no mad families in Solovay's model.
3. Schrittesser-T. (2019): If all subsets of $[\omega]^\omega$ are **completely Ramsey**, and we have "some" uniformization, then there are no mad families.

What this talk is about

The 2019 theorem localizes to pointclasses in descriptive set theory:

Loosely stated theorem (Scrittesser-T., 2019)

Let Γ be a "reasonable" pointclass with "some" uniformization (think $\Gamma = \mathbf{\Pi}^1_1$). Let $\Delta = \exists^{\mathbb{R}}\Gamma \cap \forall^{\mathbb{R}}\Gamma$ (think $\Delta = \mathbf{\Delta}^1_2$). If all Δ-sets are completely Ramsey, then there are no mad families in Γ.

In this talk, I will focus on the pointclasses Π^1_1, Σ^1_2, $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$.

Note:

- Π^1_1, Σ^1_2, $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$ all have uniformization.
- So the above theorem gives: If all Σ^1_2 sets (resp. $\mathbf{\Sigma}^1_2$ sets) are completely Ramsey, then there are no Π^1_1 (resp. $\mathbf{\Pi}^1_1$) mad families.

General theme of today's talk:

Can we **replace** the regularity property "completely Ramsey" with another regularity property and still get "no mad families"?

Remark: Because Π^1_1, Σ^1_2, $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$ have uniformization, they're a good place to investigate the above theme. *This is what we'll do in this talk.*

What is known

Very helpful fact: If there is a Σ^1_2 (resp. $\boldsymbol{\Sigma}^1_2$) mad family then there is a Π^1_1 (resp. $\boldsymbol{\Pi}^1_1$) mad family (T., 2011).

Some known theorems:

- "All $\boldsymbol{\Delta}^1_2$ sets have the **Baire Property**" does not imply "no $\boldsymbol{\Pi}^1_1$ mad families". (Essentially Kunen).
- "All $\boldsymbol{\Delta}^1_2$ sets are **Lebesgue measurable**" does not imply "no $\boldsymbol{\Pi}^1_1$ mad families". (Essentially Kunen).
- The perfect set property for $\boldsymbol{\Delta}^1_2$ sets (i.e. **Sacks measurability**) does not imply "no $\boldsymbol{\Pi}^1_1$ mad families". (Schrittesser-T.)
- The superperfect set property for $\boldsymbol{\Delta}^1_2$ sets (i.e. **Miller measurability**) does not imply "no $\boldsymbol{\Pi}^1_1$ mad families". (Schrittesser-T.)

So why does "All $\boldsymbol{\Delta}^1_2$ sets are completely Ramsey" allow us to prove there are no mad families?

The forcing perspective, I

When we want to prove a theorem such as "All Δ^1_2 sets have the **Baire Property** does not imply no Π^1_1 mad families", the the task is to produce a model of ZFC in which all Δ^1_2 sets have the Baire property and there is a Π^1_1 mad family.

Fortunately, there is a tight correspondance between adding reals using well-behaved forcing notions and measurability:

Measurability/Forcing generic real correspondance

Let x be a real, and let L be Gödel's constructible universe. Then:

- If x is a Cohen over L, then in $L[x]$ all Δ^1_2 sets have the Baire property.
- If x is a Random over L, then in $L[x]$ all Δ^1_2 sets are Lebesgue measurable.
- If x is Sacks (resp. Miller) over L, then in $L[x]$ all Δ^1_2 sets have the perfect (resp. superperfect) set property.
- If x is Mathias over L, then in $L[x]$ all Δ^1_2 sets are completely Ramsey.

The forcing perspective, II

From the forcing perspective, the theorems about **compatability / incompatability** between measurability notions and mad families translate into the following theorems.

Theorems

Let x be a real. Then

- If x is Cohen over L, then there **is** a Π^1_1 mad family in $L[x]$. (Essentially Kunen).
- If x is Random over L, then there **is** a Π^1_1 mad family in $L[x]$. (Essentially Kunen).
- If x is Sacks or Miller over L, then there **is** a Π^1_1 mad family in $L[x]$. (Schrittesser-T.)

But, by contrast, the forcing version of the theorem on the first slide says:

Theorem (Schrittesser-T., 2019)

If x is Mathias over L, then there are **no** Π^1_1 *mad families in* $L[x]$.

So why are Mathias different?

The properties of Mathias reals that make the proof of the previous theorem[1] work are:

1. **Really fast growth:** "Ramsey generic" reals grow very fast: In forcing terminology, this means that Mathias reals grow very fast. In the topological language, this means that genericity in the Ellentuck topology on $[\omega]^\omega$ ensures very fast growth.
2. **Pure decision:** In the forcing terminology, this means that we can decide a statement by only changing the infinite part of the condition. In the topological setting, it is Ellentuck's theorem.
3. **"Finite colouring homogeneity":** This is the classical infinite Ramsey theorem, which is used in an important step in the proof to refine Mathias forcing conditions/Ellentuck neighbourhoods.

[1] i.e., the theorem *"x Mathias over $L \implies$ no Π^1_1 mad families in $L[x]$"*

Where do Baire, Lebesgue, Sacks and Miller fail to work?

This list of properties (fast growth, pure decision, finite colouring homogeneity) fail for the forcing notions that produce a generic real x such that there **is** a Π^1_1 mad family in $L[x]$

- Random and Cohen reals (corresponding to Lebesgue and Baire measurability) do not grow fast enough for the proof to work, and the posets don't have pure decision.
- Sacks and Miller reals have pure decision, and Galvin's Ramsey theorems for Polish space give us a kind of colouring homogeneity that looks promising, but Sacks/Miller don't exhibit fast growth needed.

A promising poset/measurability notion?

There is a forcing notion that looks somewhat like Mathias forcing, namely **Laver forcing**. It ticks **some** of the boxes: (1) Laver reals grow very fast, (2) Laver forcing has pure decision, **but** (3) Laver forcing has poor colouring homogeneity.

Laver trees and Laver forcing

Definition (Laver trees, and the Laver poset $\mathbb{L}$)

1. A **Laver tree with stem** $s \in \omega^{<\omega}$ is a subtree $p \subseteq \omega^{<\omega}$ such that for every $t \in p$ either is an initial segment of the stem s, or it has infinitely many **immediate** successors.
2. If the stem $s = \emptyset$ then we just call p a "Laver tree".
3. The set of Laver trees with stems is denoted $\mathbb{L}$.
4. For $p, q \in \mathbb{L}$, write
$$p \leq q \iff p \subseteq q,$$
and write $p \leq^* q$ if $p \leq q$ and p and q have the **same stem**.
5. The stems of a generic filter for the poset $(\mathbb{L}, \leq)$ build a real in ω^ω, which is a Laver real (over the ground model du jour).

Remarks: (1) It should be clear that Laver reals grow really fast!

(2) Laver's poset, like Mathias' poset, has "pure decision" (Prikry property): If $q \Vdash \varphi \vee \psi$ then there is $p \leq^* q$ such that $p \Vdash \varphi$ or $p \Vdash \psi$.

Alas, Laver is no match for mad families

Despite these similarities between Laver and Mathias reals (and their posets), Laver reals can't be used to prove there are no $\mathbf{\Pi}^1_1$ mad families:

Theorem (Schrittesser-T.)

There **is** *a* Π^1_1 *mad family in* $L[r]$ *when* r *is Laver over* L.

(Of course it is enough to prove there is a Σ^1_2 mad family in $L[r]$.)

In the rest of the talk, I will discuss the main steps of the proof of this theorem, which are:

1. A corollary to a theorem due to A. Miller;
2. "Continuous reading of names for Laver reals";
3. Give a Σ^1_2 definition of a mad family in L, which remains mad in $L[r]$ when r is a Laver real; this step entails
   - A combinatorial lemma about $\mathbb{L}$-names for subsets of ω.
   - A diagonalization argument in L.

Step 1: A theorem of A. Miller's

Definition (Hechler trees)

1. A **Hechler tree with stem** $s \in \omega^{<\omega}$ is a subtree $H \subseteq \omega^{<\omega}$ such that for every $t \in H$ either is an initial segment of the stem s, or
$$\{i \in \omega : t^\frown i \in H\} \text{ is cofinite in } \omega$$
2. If the stem $s = \emptyset$ then we just call H a "Hechler tree".

Theorem (A. Miller, 2002)

(A) *If $A \subseteq \omega^\omega$ is analytic, then exactly one of the following hold:*

1. *There is a Laver tree p such that $[p] \subseteq A$ (where $[p]$ is the set of infinite branches through p);*
2. *There is a Hechler tree H such that $[H] \cap A = \emptyset$.*

(B) *If A is $\Sigma^1_1(a)$ and (2) is the case above, then there is a $\Delta^1_1(a)$ Hechler tree H witnessing this.*

Step 1: A corollary to Miller's theorem

Corollary

1. *If $A \subseteq \omega^\omega$ is an analytic set and*
$$p' \Vdash_{\mathbb{L}} x_G \in A,$$
then there is $p \leq p'$ (indeed, $p \leq^* p'$) such that $[p] \subseteq A$.
2. *If $\psi(x, y)$ is a Π^1_1 formula, then the set*
$$\{(p, a) \in \mathbb{L} \times \omega^\omega : p \Vdash_{\mathbb{L}} \psi(x_G, \check{a})\}$$
is Π^1_1.

Proof: (1) If there were no such p, then Miller's theorem gives a Hechler tree $H \leq^* p$ such that $[H] \cap A = \emptyset$, which contradicts that $H \Vdash_{\mathbb{L}} x_G \in A$.

(2) See extra slide at the end. □

Step 2: Continuous reading of names

Notation: For $y \in \omega^\omega$, we write y_i for the i'th entry of y.

Continuous reading of name is a familiar idea in forcing theory. In the case of Laver forcing, it looks like this:

Proposition (Laver, "continuous reading of names for $\mathbb{L}$")

Let τ be an $\mathbb{L}$-name (in some ground model), and suppose $p \Vdash \tau \in \omega^\omega$.

Then there is $q \leq^* p$ and a continuous function $f : [q] \to \omega^\omega$ in the ground model, such that

$$q \Vdash f(x_G) = \tau.$$

Moreover, we can arrange that whenever $q' \Vdash f(x_G)_i = j$ for some $i, j \in \omega$ and $q' \leq q$, then actually $f(x)_i = j$ for all $x \in [q]$ with $x \supseteq s(q')$.

Remark: The reason this proposition is important to us is that it gives us easy-to-work with objects (namely continous functions in L) to represent reals that will show up in the Laver extension $L[x]$.

Step 3: (A) The main lemma; (B) Diagonilization in L

Step 3.A: The main technical step of the proof is the following:

Lemma (The Main lemma)

Let $p \in \mathbb{L}$ and suppose $f : [p] \to [\omega]^\omega$ is continuous. Then there $q \leq p$ and a continuous $\tilde{f} : [q] \to [\omega]^\omega$ such that $\mathrm{ran}(\tilde{f})$ is almost disjoint and $\tilde{f}(x) \subseteq f(x)$ for all $x \in [q]$.

Step 3.B: Proving the theorem (repeated here for convenience):

Theorem (Schrittesser-T.)

There **is** *a Π^1_1 mad family in $L[r]$ when r is Laver over L.*

Given the Main Lemma, the proof of the theorem is a fairly standard diagonalization argument, *which I will sketch on the next slides.*

Notation: $C(\omega^\omega, [\omega]^\omega)$ denotes the set of continuous functions from ω^ω to $[\omega]^\omega$. By continuous reading of names, $C(\omega^\omega, [\omega]^\omega)$ is morally the set of all $\mathbb{L}$-names for infinite subsets of ω.

Step 3.B: Diagonilization in L

Work in L. We will construct a family $(A_\xi)_{\xi<\omega_1}$ of $\mathbf{\Sigma}^1_1$ almost disjoint families such that $\bigcup_{\xi<\omega_1} A_\xi$ will be Σ^1_2 and mad, even in $L[r]$. (r Laver).

- First fix an enumeration of $\mathbb{L} \times C(\omega^\omega, [\omega]^\omega)$, call it $(p_\xi, f_\xi)_{\xi\in\omega_1}$, which corresponds to a good Σ^1_2 well-ordering of $\mathbb{L} \times C(\omega^\omega, [\omega]^\omega)$.
- Let A_0 be any infinite Σ^1_1 almost disjoint family.
- Assume A_γ have been defined for $\gamma < \xi$, and ask if

$$p_\xi \Vdash (\forall\gamma < \xi)(\forall y \in \mathcal{A}_\gamma)|f_\xi(x_G) \cap y| < \infty. \tag{1}$$

  - **If yes**, apply the main lemma to (p, f) to get $(q, \tilde{f})$, **which we can assume is the $\leq_L$-least such** $(q, \tilde{f})$, and let

$$A_\xi = \operatorname{ran}(\tilde{f}) \cup \bigcup_{\gamma<\xi} A_\gamma.$$

  - **If no,** just let $A_\xi = \bigcup_{\gamma<\xi} A_\gamma$.
- Due to the uniformity of the construction, and the fact that checking (1) is Π^1_1 in the parameters by Step 1, there is a natural Σ^1_2 predicate $\varphi(x)$ asserting $(\exists\xi < \omega_1)\ x \in A_\xi$.

$$\boxed{\textbf{We claim that } \varphi \textbf{ defines a mad family in } L[r].}$$

Step 3.B: Diagonilization in L (cont'd)

To see that φ defines a mad family in $L[r]$, suppose for a contradiction this is not the case, and argue as follows:

- In this situation, there must be a name τ for an infinite subset of ω and $p \in \mathbb{L}$ such that

$$p \Vdash (\forall x)\ \varphi(x) \to |\tau \cap x| < \infty.$$

- By continuous reading of names, we can then find a continuous $f : \omega^\omega \to [\omega]^\omega$ in L such that

$$p \Vdash (\forall x)\ \varphi(x) \to |f(x_G) \cap x| < \infty. \tag{2}$$

- There must be some ξ such that $(p_\xi, f_\xi) = (p, f)$.
- By Eq. (2) we get

$$p_\xi \Vdash (\forall \gamma < \xi)(\forall x)\ x \in A_\gamma \to |f_\xi(x_G) \cap x| < \infty.$$

- This means that the the answer at stage ξ of the construction is **yes**.
- This contradicts that we made sure there is $q \leq p_\xi = p$ such that

$$q \Vdash (\exists x \in A_\xi)\ |f(x_G) \cap x| = \infty. \quad \square$$

Step 3.A: The Main Lemma

Lemma (The Main lemma)

Let $p \in \mathbb{L}$ and suppose $f : [p] \to [\omega]^\omega$ is continuous. Then there $q \leq p$ and a continuous $\tilde{f} : [q] \to [\omega]^\omega$ such that $\mathrm{ran}(\tilde{f})$ is almost disjoint and $\tilde{f}(x) \subseteq f(x)$ for all $x \in [q]$.

This is quite a technical lemma to prove, so I will only say a little, and only about what the overall idea is.

- First, fix p and f as in the Lemma.
- Next, it is useful now to identify $[\omega]^\omega$ with the strictly increasing elements of ω^ω. *So we do that.*
- For $p \in \mathbb{L}$ and $t \in p$, write p/t for the notes in p compatible with p (this is then a Laver tree with a stem that extends t).
- For $i \in \omega$ and $t \in p$, we ask if the value of $f(x_G)_i$ can be decides "locally" at t, that is, if there is $q \leq^* p/t$ and $j \in \omega$ such that

$$q \Vdash f(x_G)_i = j.$$

Step 3.A: The Main Lemma (cont'd)

The point is now:

- At $t \in p$ we can decide $f(x_G)_i$ locally for all i, then we have a lot of control over the values of $f(x)$ for $x \in [p/t]$, so the desired $\tilde{f}$ is quite easy to construct.
- At $t \in p$ where there is $i \in \omega$ such that $f(x_G)_i$ cannot be decided locally, we have enourmous freedom to make $f(x_G)_i$ as large as we want by going further out into the Laver tree (in many different directions).
- This fact can then be used to build $\tilde{f}$ in this case.
- The gory details will be in the paper[2].

[2] a very poorly written version with endless typos exists right now, but sometime in a week or two there will be a cleaned up version. Please email me in a week or two.

Some questions to end with

- **Question:** There are myriad other forcing notions that are "tree-like" where the question if $L[r]$ for such a real has a Π^1_1 mad family. For instance, there is Brendle's Willowtree forcing, there is Silver forcing ("doughnut forcing"), etc., etc. I don't know what the situation is for these forcing notions.
- **Question:** People have been trying for at least 10 years to prove Mathias' theorem "There are no analytic mad families" using the G_0 dichotomy, but as far as I know no one has succeeded. Is there a reason for this? Is the fact that the Ramsey property works as a genericity property to prove this fact, but Baire, Lebesgue, Sacks, Miller, Laver don't, a hint that it is actually not possible[3]
- **Question:** Does AD imply there are no mad families?

[3] I know that any two true statemens imply each other. I also know when Christmas is.

Thanks for listening!

Proof of (2) of the Corollary on slide 12

Proof: (2) Fix $a \in \omega^\omega$, and consider the set

$$A_{a,p} = \{x \in [p] : \neg\psi(x, a)\},$$

which is a $\Sigma^1_1(a)$ set (since ψ is a Π^1_1 predicate).

Claim: $p \Vdash \psi(x_G, \check{a})$ if and only if there exists a stemmed Hechler tree $H \in \Delta^1_1(a, p)$ with $H \leq^* p$ and $[H] \cap A_{a,p} = \emptyset$.

<u>*Proof of Claim:*</u> If there is no such H, then by the effective version of Miller's theorem, there is $q \leq^* p$ such that $[q] \subseteq A_{a,p}$. Then $q \Vdash \neg\psi(x_G, \check{a})$ (by a Shoenfield absoluteness argument), so $p \nVdash \psi(x_G, \check{a})$.

Conversely, suppose $H \in \Delta^1_1(a)$ is such that $[H] \cap A_{a,p} = \emptyset$. Then $H \Vdash \psi(x_G, a)$ by a Shoenfield absoluteness argument. To see that $p \Vdash \psi(\dot{x}_G, \check{a})$, let $p' \leq p$. Then $p' \cap H$ is a stemmed Laver tree and $p' \cap H \leq^* p'$ <u>and</u> $p' \cap H \leq H$. So $p' \cap H \Vdash \psi(x_G, a)$. (Claim)$\square$

Now we're done, since

$$p \Vdash \psi(x_G, \check{a}) \iff (\exists H \in \Delta^1_1(p, a) \text{ with } H \leq^* p)(\forall x \in H)\ \psi(x, a)$$

is the required Π^1_1 definition by Spector-Gandy. $\square$