Bowen's Problem 32 and the conjugacy problem for systems with specification

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Structure of the talk

1. **Bowen's Problem 32** and conjugacy of symbolic systems with specification
2. Conjugacy of Cantor systems with specification and a **problem of Ding and Gu**,
3. Conjugacy of Hilbert cube systems with specification and a **problem of Bruin and Vejnar**.

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The problem is currently maintained on a dedicated website
`https://bowen.pims.math.ca/problems/32`

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Theorem (**Sabok**)

The classification problem for **separable C^*-algebras** is a **complete orbit equivalence relation**.

Theorem (**Zielinski**)

The classification problem for **compact metrizable spaces** is a **complete orbit equivalence relation**.

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1. The conjugacy relation of **measure preserving transformations**.
2. **Topological conjugacy** of homeomorphisms (diffeomorphisms).

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Theorem (Foreman-Gorodetski)

Let M be a manifold of dimension n, then the topological conjugacy relation of **smooth diffeomorphisms** on M is

1. **not smooth** when $n \geq 2$,
2. **not Borel** when $n \geq 5$.

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Subshifts

A **symbolic system** is a subsystem of the shift $(\{0,1\}^{\mathbb{Z}}, \sigma)$.

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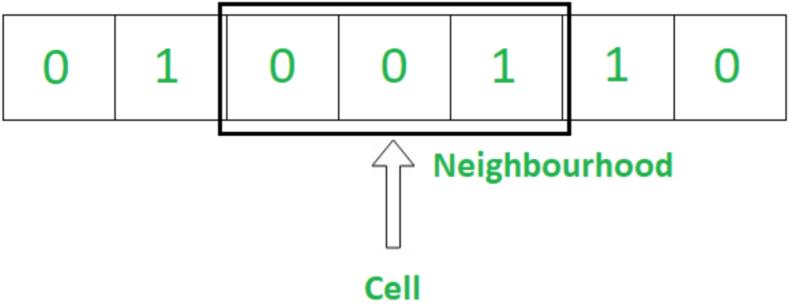

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Theorem (Curtis–Hedlund–Lyndon)

Any continuous shift-equivariant map between symbolic systems is given by a **block code**.

Conjugacy in symbolic dynamics

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Countable Borel equivalence relation

Note that since there are only countably many block codes, the conjugacy is symbolic dynamics is a **countable Borel equivalence relation**.

Transitive point

A point in a system is **transitive** if its orbit is dense.

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Transitive systems

A system is **transitive** if it has a transitive point. A system is **minimal** if every point is transitive.

$=^+$ is an equivalence relation defined on $\mathbb{R}^\omega$:

$$(x_n) =^+ (y_n) \text{ if and only if } \{x_n\} = \{y_n\}.$$

Pointed systems

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Theorem (Kaya)

The conjugacy relation of **pointed Cantor minimal systems** is Borel bi-reducible with $=^+$.

Orbit segment

Given a map $\tau\colon X \to X$, an interval $[a, b] \subseteq \mathbb{N}$ with $0 \leq a < b$, and $x \in X$ we write

$$\tau^{[a,b]}(x)$$

for the sequence $(\tau^i(x))_{a \leq i \leq b}$ and call it the **orbit segment** (of x over $[a, b]$).

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k-spaced specification

Let k be a natural number. A **k-spaced specification** is a family of orbit segments

$$\{\tau^{[a_i,b_i]}(x_i) : 1 \le i \le n\}$$

such that

$$a_i - b_{i-1} > k \quad \text{for} \quad 2 \le i \le n.$$

Tracing orbit segments

Fix $\varepsilon > 0$. A specification $\{\tau^{[a_i,b_i]}(x_i) : 1 \leq i \leq n\}$ is **ε-traced** if there **exists** $y \in X$ such that

$$d(\tau^j(x_i), \tau^j(y)) \leq \varepsilon$$

for $j \in [a_i, b_i]$, for every $1 \leq i \leq n$.

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$$\begin{array}{cccc} \tau^{a_1}(\boldsymbol{x_1})\dots\tau^{b_1}(\boldsymbol{x_1}) & \tau^{a_2}(\boldsymbol{x_2})\dots\tau^{b_2}(\boldsymbol{x_2}) & \dots & \tau^{a_n}(\boldsymbol{x_n})\dots\tau^{b_n}(\boldsymbol{x_n}) \\ \tau^{a_1}(\boldsymbol{y})\dots\tau^{b_1}(\boldsymbol{y}) & \tau^{a_2}(\boldsymbol{y})\dots,\tau^{b_2}(\boldsymbol{y}) & \dots & \tau^{a_n}(\boldsymbol{y})\dots\tau^{b_n}(\boldsymbol{y}) \end{array}$$

Definition

A dynamical system (X, τ) has the **specification property** if for every $\varepsilon > 0$ there **exists** $k(\varepsilon) \in \mathbb{N}$ such that every $k(\varepsilon)$-spaced specification **is ε-traced by a point** from X.

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This terminology is **classical** but in more modern terminology, **equivalent** notion (also, more more general group actions than $\mathbb{Z}$) goes under the name **strong irreducibility** (e.g. in the works of **Frisch, Seward, Tsankov or Zucker**).

Theorem (**Thomas, Gao–Jackson–Seward**, independently)

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It is **not known** whether the conjugacy relation for minimal symbolic systems is **hyperfinite or not**.

Language

Given a symbolic system $X \subseteq A^{\mathbb{Z}}$, its **language** is the collection of finite words appearing in its elements:

$$\mathrm{Lang}(X) := \{x|[i,j] \colon x \in X \text{ and } i \leq j\}.$$

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Fact

Transitivity of a symbolic system X is **equivalent** to the statement that for all $\boldsymbol{u}, \boldsymbol{v} \in \mathrm{Lang}(X)$, there exists $\boldsymbol{w}$ such that

$$\boldsymbol{uwv} \in \mathrm{Lang}(X)$$

Specification

A symbolic system X has the **specification property** if and only if there exists a natural number k such that for all $u, v \in \mathrm{Lang}(X)$, there exists w such that

$$|w| = k$$

and

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and

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Note that specification **implies** transitivity.

Bowen's Problem 32

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The reason following result explain why such classification is in fact **impossible**.

Theorem (Deka–Kwietniak–P.–Sabok)

The conjugacy of symbolic systems with specification is **not smooth**.

In fact, we prove a somewhat stronger statement.

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Theorem (Deka–Kwietniak–P.–Sabok)

The conjugacy of symbolic systems with specification is **not hyperfinite**.

The proof consists of constructing a specific class of symbolic systems with the specification property

Fact

There exists a class of symbolic systems with the specification property that admits a **probability measure** an an **action of the free group F_2** which preserves conjugacy and the probability measure such that the action is a.e. free.

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This implies that the conjugacy restricted to the class is not hyperfinite. In fact, the same proof can be also used to further strengthen the non-hyperfiniteness.

The construction of the symbolic systems with specification is purely **combinatorial**.

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It relies on **counting the number of occurences of certain blocks** of 0's and 1's.

To make sure that the systems have the specification property we need to use blocks whose lenghts are **relatively prime**.

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For instance such blocks of lengths 7 and 5 work:

$$\begin{aligned} a &= 1000001, \\ b &= 1001001, \\ c &= 1011101, \\ d &= 1010101, \\ \# &= 10001. \end{aligned}$$

Question

Is the conjugacy relation for symbolic systems with specification a universal countable Borel equivalence relation?

Let us look at something **unrelated** to dynamical systems.

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Given two **metrics** d_1 and d_2 on $\mathbb{N}$ we write

$$d_1 \; E_{\mathrm{sc}} \; d_2$$

if the identity function $n \mapsto n$ on $\mathbb{N}$ extends to a **homeomorphism of the completions** of d_1 and d_2.

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In their paper, Ding and Gu **ask what is the complexity** of E_{sc} **restricted to metrics whose completion is compact zero-dimensional**.

In fact, they ask a specific question about the **upper bound on the above complexity** that is weaker but the answer to the above will answer their question as well.

How is this related to systems with specification?

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Fact

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The conjugacy problem for **pointed Cantor systems with specification** is Borel-reducible to E_{sc} **restricted to metrics whose completion is compact zero-dimensional**.

Proof

Given a pointed Cantor system with specification (X, φ, x), map it to the metric on $\mathbb{N}$ induced on via the map $n \mapsto \varphi^n(x)$. That is put

$$d(n, m) = d_X(\varphi^n(x), \varphi^m(x)).$$

This is a Borel reduction.

By the way of computing the complexity of the conjugacy of pointed Cantor systems with specification, **we solve Ding and Gu's problem**.

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Theorem (Deka–Kwietniak–P.–Sabok)

The E_{sc} **restricted to metrics whose completion is compact zero-dimensional** is $=^{+}$.

The proof actually uses dynamical systems and a notion of the so-called **Oxtoby systems** originating in **work of Williams** from 1980's, as well as a result of **Kaya**.

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Corollary

The **complexity of the conjugacy of pointed Cantor systems with specification** is $=^+$.

A **Hilbert cube system** is a system whose underlying space is the Hilbert cube

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A very similar question to Bowen's problem but for pointed systems was also asked in a recent paper of Bruin and Vejnar.

Table from the paper of Bruin–Vejnar

	pointed transitive systems.
interval	$\emptyset$
circle	$=$
Cantor set	$=^{+}$ (Kaya)
Hilbert cube	**?**

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Question (Bruin–Vejnar)

What is the complexity of the conjugacy of **pointed transitive Hilbert cube systems**?

Fact

The following relations have the same complexity

- conjugacy of pointed **transitive** Hilbert cube systems
- conjugacy of pointed Hilbert cube systems **with the specification property**
- the action of $\mathrm{Aut}((([0,1]^{\mathbb{N}})^{\mathbb{Z}}, \sigma))$ on the set

$$\{x \in ([0,1]^{\mathbb{N}})^{\mathbb{Z}} : x \text{ is } \textbf{transitive}\}$$

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Corollary

The conjugacy of pointed transitive Hilbert cube systems is bi-reducible with a **turbulent group action**.