Sigma Scattered Linear Ordering

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The Descriptive Set Theoretic World

The descriptive set theoretic world is an idealized choiceless universe for uniform mathematics. Definable behavior from classical descriptive set theory and Borel combinatorics extend to genuine combinatorial properties.

The axiom of determinacy, AD, seems to be among one of the laws which are accepted to govern the initial segment of this descriptive set theoretic universe of sets which are images of $\mathbb{R}$.

The descriptive set theoretic universe with the determinacy axiom is immune to the method of forcing by posets which are images of $\mathbb{R}$ (C.-Jackson, Ikegami-Trang). Empirically, it seems very successful at settling the basic questions of interests to set theory. Two classical objects of interests to set theorists are mathematical size (cardinality) and linear orderings.

The classical dichotomies of Silver and Harrington-Kechris-Louveau proved by Gandy-Harrington forcing can be extended to general dichotomies concerning cardinalities and linear orderings under Woodin's theory of AD^+ using ordinal definable ∞-Borel code forcing (a variety of Vopěnka forcing).

Theorem (Woodin; Perfect set dichotomy)
(AD^+) *Let X be an image of $\mathbb{R}$. Exactly one of the following holds.*

- *X is wellorderable.*
- *$|\mathbb{R}| \leq |X|$ (and hence X is not wellorderable).*

Theorem (Hjorth; E_0-dichotomy)
(AD^+). *Let X be an image of $\mathbb{R}$. Exactly one of the following holds.*

- *$|X| \leq |\mathscr{P}(\kappa)|$ for some ordinal κ (and hence X is linearly orderable).*
- *$|\mathbb{R}/E_0| \leq |X|$ (and hence X is not linearly orderable).*

The important examples on the non-linearly orderable side of Hjorth's dichotomy (above $\mathbb{R}/E_0$) are often quotients of familiar Borel equivalence relations. The full structural results tend to be generalization of the classical Borel argument using the regularity properties of determinacy.

The descriptive set theoretic investigation of the linearly orderable side of Hjorth's dichotomy (subsets of the power set of ordinals) have received less attention. Here the theory much more robust under full determinacy rather than local definability restrictions.

One recent result concerning cardinality on the linearly orderable side is the classification of cardinal exponentiation. Θ is the supremum of the ordinals which are images of $\mathbb{R}$.

Theorem (C.; ABCD Conjecture)
(AD^+) *If* $\omega \leq \alpha < \beta < \Theta$ *and* $\omega \leq \gamma \leq \delta < \Theta$ *are cardinals, then* $|{}^\alpha \beta| \leq |{}^\gamma \delta|$ *if and only if* $\alpha \leq \gamma$ *and* $\beta \leq \delta$*.*

Exotic Linear Orderings

Set theory within the choiceful framework (ZFC and its extensions) have investigated some exotic linear orderings including Suslin lines, Aronszajn lines, and Baumgartner lines. Forcing axioms such as PFA have had great success in establishing structural results for linear orderings.

Effort to study linear orderings in the choiceless descriptive set theoretic framework have been motivated by showing these exotic linear orderings do not exists. This may not seem particular productive, but the proofs usually are a consequence of remarkable structural consequences for linear orderings under determinacy hypothesis.

Determinacy provides a powerful framework for investigating linear orderings which are images of $\mathbb{R}$.

Cantor showed that every complete separable linear ordering without endpoint is isomorphic to the usual ordering on $\mathbb{R}$. The Suslin problem asks if this is still true if separability is weakened to the countable chain condition. An counterexample is a called a Suslin line.

Theorem (Chan-Jackson)
Assume AD^+*. There are no Suslin lines which are images of* $\mathbb{R}$*.*

This is a consequence of a more interesting structural result called the prelinear ordering dichotomy proved using the ordinal definable ∞-Borel code forcing.

An Aronszajn line is a linear ordering which does not contain a suborder isomorphic to ω_1, ω_1^*, or an uncountable subordering of the usual ordering on $\mathbb{R}$.

The more productive reformulation of the nonexistence of Aronszajn line involves basis for linear orderings.

Theorem (Weinert)
Assume AD^+. $\{\mathbb{R}, \omega_1, \omega_1^*\}$ *form a three element basis for the the class of uncountable linear ordering which are images of* $\mathbb{R}$*. Hence there are no Aronszajn line on an image of* $\mathbb{R}$*.*

Proof.
Suppose $(X, \prec)$ is a linear ordering with X uncountable. By the Woodin perfect set dichtomy, X must have an injective copy of ω_1 or an injective copy of $\mathbb{R}$. Since ω_1 satisfies the exponent two partition relation, ω_1 or ω^* order embeds into $(X, \prec)$ if ω_1 injects into X. Since $\mathbb{R}$ satisfies a perfect tree partition relation, the usual ordering on $\mathbb{R}$ embeds into $(X, \prec)$ if $\mathbb{R}$ injects into X. □

Basis for Uncountable Linear Orderings

(Chan) Assume AD^+.

- There is a two-element basis for the class of linear orderings whose cardinality is greater than or equal to κ when $\kappa < \Theta$ is an uncountable regular cardinal.
- There is a four-element basis for the class of all linear orderings whose cardinality is greater than or equal to $|\mathbb{R} \sqcup \kappa|$ when $\kappa < \Theta$ is an uncountable regular cardinal.
- There is a four-element basis for the class of all linear ordering whose cardinality is greater than or equal to $|\mathbb{R} \times \kappa|$ where $\kappa < \Theta$ is an uncountable regular cardinal.
- There is a four-element basis for the class of all linear orderings whose cardinality is greater than or equal to κ when $\kappa < \Theta$ is a singular cardinal.
- There is an eight-element basis for the class of all linear orderings whose cardinality is greater than or equal to $|\mathbb{R} \sqcup \kappa|$ when $\kappa < \Theta$ is a singular cardinal.
- There is a twelve-element basis for the class of all linear orderings whose cardinality is greater than or equal to $|\mathbb{R} \times \kappa|$ when $\kappa < \Theta$ is a singular cardinal of uncountable cofinality.
- There is a four-element basis for the class of all linear orderings whose cardinality is greater than or equal to $|{}^{\omega}\kappa|$ when κ satisfies $\kappa \rightarrow_* (\kappa)^{\omega+\omega}_2$.
- There is a two element basis for the class of all linear orderings whose cardinality is greater than or equal to $|{}^{\omega}\kappa|$ when $\omega < \kappa < \Theta$ and $\mathrm{cof}(\kappa) = \omega$.

This talk was motivated by whether Baumgartner lines exist in the descriptive set theoretic universe.

Definition
A linear ordering $\mathcal{L} = (L, \prec)$ is scattered if the usual ordering $\mathbb{Q}$ on the rationals do not order embed into $\mathcal{L}$. $\mathcal{L}$ is σ-scattered if and only if there is a family $\langle A_n : n \in \omega \rangle$ so that $\mathcal{L} \upharpoonright A_n$ is scattered for each $n \in \omega$ and $L = \bigcup_{n \in \omega} A_n$.

Definition
A Baumgartner line is a non-σ-scattered linear ordering which does not have an order isomorphic copy of $\mathbb{R}$ or an Aronszajn line.

Baumgartner constructed a Baumgartner line under ZFC by using a stationary $S \subseteq \omega_1$ and a sequence $\langle f_\alpha : \alpha \in S \rangle$ so that $f_\alpha : \omega \to \alpha$ is cofinal. Under AD, the partition properties on ω_1 implies that the club filter on ω_1 is a normal measure and thus such objects cannot exist.

By the partition property on $\mathbb{R}$ and Woodin perfect set dichotomy, not having even an injective copy of $\mathbb{R}$ means the linear ordering must be on wellorderable set. So a potential Baumgartner line is nothing more than a non-σ-scattered linear ordering on a wellorderable set (or equivalently an ordinal).

Sigma Scattered Linear Orderings

The usual ordering on $\mathbb{R}$ is a non-σ-scattered linear ordering. Whenever a linear ordering is on a unwellorderable set, it has an injective copy of $\mathbb{R}$ and thus a subordering isomorphic to $\mathbb{R}$. Hence it is non-σ-scattered.

Is it really possible to create a non-σ-scattered linear ordering that does not have the usual $\mathbb{R}$ as a subordering using descriptive set theoretically accepted uniform methods?

An answer of no in the presence of AD^+ gives a very nice characterization of σ-scattered linear orderings: Let $\mathcal{L} = (L, \prec)$ be a linear ordering which is a image of $\mathbb{R}$. The following are equivalent.

1. $\mathcal{L}$ is σ-scattered.
2. There is no order embedding of usual $\mathbb{R}$ into $\mathcal{L}$.
3. There is no injection of $\mathbb{R}$ into L.
4. L is a wellorderable set.

Fraïssé's Conjecture and Laver's Theorem

Fact (Laver)
The class of σ-scattered linear ordering is a well quasiordering under order embeddings: Every sequence $\langle \mathcal{L}_i : i \in \omega \rangle$ consisting of σ-scattered linear ordering, then there exists $i < j < \omega$ so that $\mathcal{L}_i$ order embeds into $\mathcal{L}_j$.

Laver's theorem solves the Fraïssé conjecture which states the class of countable linear ordering (the linear ordering which are images of ω) form a well quasiordering under order embeddings.

In recent conversion with Moore and Todorcevic, they mentioned a classical question of whether the class of Borel linear orderings form a well quasiordering under order embeddings. Moore suggested the following descriptive set theoretic Fraïssé conjecture: Under determinacy assumptions, does the class of all linear orderings which are images of $\mathbb{R}$ form a well quasiorder under order embeddings?

Woodin's perfect set dichotomy splits the linearly orderable sets further into the wellorderable sets and those sets which have a copy of of $\mathbb{R}$. If the characterization of σ-scattered linear orderings hold, then Laver's theorem can be extended to prove the wellorderable Fraïssé conjecture for the class of linear orderings on wellorderable sets which are images of $\mathbb{R}$ (or linear orderings which are images of an ordinal below Θ).

Every linear ordering which is an image of ω is σ-scattered since every countable linear ordering is σ-scattered. We will focus on linear orderings on the first uncountable cardinals ω_1.

We will need the correct type partition relation on ω_1 since we will need to analyze the partition measures.

Definition
Let $\epsilon \in \mathrm{ON}$. A function $f : \epsilon \to \mathrm{ON}$ has the correct type if and only if the following holds.

- f is discontinuous everywhere: For all $\alpha < \epsilon$, $\sup(f \upharpoonright \alpha) < f(\alpha)$.
- f has uniform cofinality ω: There is a function $F : \epsilon \times \omega \to \mathrm{ON}$ so that for all $\alpha < \epsilon$ and $n \in \omega$, $F(\alpha, n) < F(\alpha, n+1)$ and $f(\alpha) = \sup\{F(\alpha, n) : n \in \omega\}$.

If $A \subseteq \mathrm{ON}$, then $[A]^\epsilon_*$ is the set of a increasing $f : \epsilon \to A$ which have the correct type.

Definition
If $\epsilon \leq \kappa$ and $\gamma < \kappa$, then let $\kappa \to_* (\kappa)^\epsilon_\gamma$ be the assertion that for all $P : [\kappa]^\epsilon_* \to \gamma$, there is a $\beta < \gamma$ and a club $C \subseteq \kappa$ so that for all $f \in [C]^\epsilon_*$, $P(f) = \beta$.

Definition
If $\epsilon \leq \kappa$, then let μ_κ^ϵ be a filter on $[\kappa]^\epsilon$ defined by $A \in \mu_\kappa^\epsilon$ if and only if there is a club C so that $[C]_*^\epsilon \subseteq A$.

Fact
If $\epsilon \leq \kappa$, $\gamma < \kappa$, and $\kappa \to_* (\kappa)_\gamma^\epsilon$, then μ_κ^ϵ is a γ^+-complete ultrafilter on $[\kappa]_*^\epsilon$.

If $\kappa \to_* (\kappa)_2^2$ holds, then the ω-club filter μ_κ^1 is a κ-complete normal ultrafilter on κ.

Martin showed that under AD, ω_1 satisfies $\omega_1 \to_* (\omega_1)_{<\omega_1}^{\omega_1}$ or is sometimes called a very strong partition cardinals. We will primarily be interested in the measures $\mu_{\omega_1}^n$, the n-fold product of the club filter on ω_1.

First, one will define some canonical linear orderings on $[\omega_1]^n$ for each $1 \leq n < \omega$. $\ell \in [\omega_1]^n$ will be regared as an increasing function $\ell : n \to \omega_1$.

If $\mathcal{X} = (X, \prec)$, then $\prec^0$ denote $\prec$ and $\prec^1$ denote the reverse of $\prec$.

Definition
For $1 \leq n < \omega$, let Bij_n be the set of all bijections $\rho : n \to n$. Let ${}^n2$ be a set of function from n into 2. Fix $\tau \in {}^n2$ and $\rho \in \mathrm{Bij}_n$. Define the linear ordering $\mathcal{L}^{n,\tau,\rho} = ([\omega_1]^n, \prec^{n,\tau,\rho})$ defined by $\ell_0 \prec^{n,\tau,\rho} \ell_1$ if and only if $\ell_0 \neq \ell_1$ and if m is the least m' such that $\ell_0(\rho(m')) \neq \ell_1(\rho(m'))$, then $\ell_0(\rho(m)) <^{\tau(m)} \ell_1(\rho(m))$.

The function τ determines the direction of each coordinate: the m^{th} coordinate is ordered by $<^{\tau(m)}$. The bijection ρ is a ranking of the coordinates in decreasing strength. Coordinate $\rho(0)$ is strongest and is compared first according to $<^{\tau(\rho(0))}$. Then coordinate $\rho(1)$ is compared next using $<^{\tau(\rho(1))}$. And so on ...

Example
Let $n = 1$. ${}^1 2 = \{(0), (1)\}$. $\mathrm{Bij}_1 = \{(0)\}$.

- $\mathcal{L}^{1,(0),(0)} = (\omega_1, <^0) = (\omega_1, <)$.
- $\mathcal{L}^{1,(1),(0)} = (\omega_1, <^1)$, the reverse ω_1.

Note that both linear orderings are scattered.

Example
Let $n = 2$. ${}^2 2 = \{(0,0),(1,1),(0,1),(1,0)\}$. $\mathrm{Bij}_2 = \{(0,1),(1,0)\}$.

- $\mathcal{L}^{2,(0,0),(0,1)}$ is order isomorphic to $\prod_{\omega_1} \omega_1$.
- $\mathcal{L}^{2,(0,0),(1,0)}$ is order isomorphic to ω_1.
- $\mathcal{L}^{2,(0,1),(0,1)}$ is order isomorphic to $\prod_{\omega_1} {\omega_1}^*$.
- $\mathcal{L}^{2,(0,1),(1,0)}$ is order isomorphic to $\prod_{\alpha \in {\omega_1}^*} \alpha$.
- $\mathcal{L}^{2,(1,0),(0,1)}$ is order isomorphic to $\prod_{{\omega_1}^*} \omega_1$.
- $\mathcal{L}^{2,(1,0),(1,0)}$ is order isomorphic to $\prod_{\alpha \in \omega_1} \alpha^*$.
- $\mathcal{L}^{2,(1,1),(0,1)}$ is order isomorphic to $\prod_{{\omega_1}^*} {\omega_1}^*$.
- $\mathcal{L}^{2,(1,1),(1,0)}$ is order isomorphic to ${\omega_1}^*$.

Each of these eight linear orderings are scattered since they are wellordered or reverse wellordered lexicographic product of scattered linear orderings.

By induction, one can show all $\mathcal{L}^{n,\tau,\rho}$ are wellordered or reverse wellordered lexicographic product of scattered linear orderings.

Fact
For all $1 \leq n < \omega$, $\tau \in {}^n2$, and $\rho \in \mathrm{Bij}_n$. $\mathcal{L}^{n,\tau,\rho}$ is scattered.

Theorem
Assume $\omega_1 \to_* (\omega_1)^{n+2}_2$. *Let* $1 \leq n < \omega$ *and* $\mathcal{L} = ([\omega_1]^n, \prec)$ *is a linear ordering on* $[\omega_1]^n$. *There is a club* $C \subseteq \omega_1$, $\tau \in {}^n2$, *and* $\rho \in \mathrm{Bij}_n$ *so that* $\mathcal{L} \restriction [C]^n = \mathcal{L}^{n,\tau,\rho} \restriction [C]^n$.

This is the main combinatorial result. We will return to sketch a proof of this result.

For our purpose, a measure is a countably complete ultrafilter. Under AD, a measure is simply an ultrafilter since all ultrafilters are countably complete because there are no nonprincipal ultrafilter on ω.

Definition
Let X and Y be two sets. Let $\Phi : X \to Y$. Let $\mathcal{F}$ be a filter on X. The Rudin-Keisler pushforward of $\mathcal{F}$ by Φ is the filter $\Phi_*\mathcal{F}$ on Y defined by $B \in \Phi_*\mathcal{F}$ if and only if $\Phi^{-1}[B] \in \mathcal{F}$.

If $\mathcal{F}$ is a filter on X and $A \in \mathcal{F}$, then let $\mathcal{F} \restriction A$ be the filter on A defined by $A' \in \mathcal{F} \restriction A$ if and only if $A' \subseteq A$ and $A' \in \mathcal{F}$.

Let $\mathcal{F}$ be a filter on X and $\mathcal{G}$ be a filter on Y. $\mathcal{F}$ and $\mathcal{G}$ are Rudin-Keisler equivalent if and only if there are $A \in \mathcal{F}$, $B \in \mathcal{G}$, and a bijection $\Phi : A \to B$ so that $\mathcal{G} \restriction B = \Phi_*(\mathcal{F} \restriction A)$ and $\mathcal{F} \restriction A = (\Phi^{-1})_*(\mathcal{G} \restriction B)$.

Fact (Kunen)
Assume AD *and* $\mathsf{DC}_{\mathbb{R}}$. *Let μ be a measure on ω_1. There is an $n \in \omega$ so that μ is Ruden-Keisler equivalent to $\mu^n_{\omega_1}$. (Note that $\mu^0_{\omega_1}$ is a principal measure.)*

Definition
Let X be a set. Let μ be a measure on X. Let $\mathscr{B}$ be a set of linear orderings. $\mathscr{B}$ is a biembedability basis for μ if and only if for any linear ordering $\mathcal{L} = (X, \prec)$, there is an $A \in \mu$ and a $\mathcal{J} \in \mathscr{B}$ so that $\mathcal{L} \upharpoonright A$ and $\mathcal{J}$ are biembeddable.

Theorem
Assume AD *and* $\mathsf{DC}_{\mathbb{R}}$. *Let μ be a nonprincipal measure on ω_1. Then there is a $1 \leq n < \omega$ so that $\{\mathcal{L}^{n,\tau,\rho} : \tau \in {}^n 2 \wedge \rho \in \mathrm{Bij}_n\}$ is a biembeddability basis for μ consisting of $n!2^n$ scattered linear orderings.*

Proof.
By Kunen's theorem, there is a $1 \leq n < \omega$, $A \in \mu$, $B \in \mu^n_{\omega_1}$, and a bijection $\Phi : A \to B$ so that $\mu^n_{\omega_1} \upharpoonright B = \Phi_*(\mu \upharpoonright A)$ and $\mu \upharpoonright A = (\Phi^{-1})_*(\mu^n_{\omega_1} \upharpoonright B)$. Let $\mathcal{L} = (\omega_1, \prec)$ be a linear ordering. Let $(B, \sqsubset)$ be defined by $\ell_0 \sqsubset \ell_1$ if and only if $\Phi^{-1}(\ell_0) \prec \Phi^{-1}(\ell_1)$. By the previous fact, there is a club $C \subseteq \omega_1$, $\tau \in {}^n 2$, and $\rho \in \mathrm{Bij}_n$ so that $[C]^n \subseteq B$ and $([C]^n, \sqsubset) = \mathcal{L}^{n,\rho,\tau} \upharpoonright [C]^n$. This shows $\mathcal{L} \upharpoonright A$ and $\mathcal{L}^{n,\tau,\rho}$ are biembeddable. $\square$

Theorem
Assume AD *and* $\mathsf{DC}_{\mathbb{R}}$. *Let $\mathcal{L} = (\omega_1, \prec)$ be a linear ordering on ω_1 and let μ be a measure on ω_1. Then there is an $A \in \mu$ so that $\mathcal{L} \upharpoonright A$ is scattered.*

Theorem
Assume AD. *Every linear ordering on ω_1 is σ-scattered.*

Suppose the result fails. Let $\mathcal{L} = (\omega_1, \prec)$ be a non-σ-scattered linear ordering on ω_1. All the previous results used $\mathsf{DC}_{\mathbb{R}}$ so first let us get into a situation in which $\mathsf{DC}_{\mathbb{R}}$ holds.

Let $\mathrm{ot} : \mathrm{WO} \to \omega_1$ be the Π^1_1 norm of length ω_1 given by the order type function. By the Moschovakis coding lemma, there is a surjection $\pi : \mathbb{R} \to \mathscr{P}(\omega_1)$ which is "$\boldsymbol{\Sigma}^1_2$ in some sense".

By using π and Moschovakis coding lemma, $\mathcal{L}$ and in fact every subset of ω_1 in the real world already belongs to the inner model $L(\mathbb{R})$. For any $A \in \mathscr{P}(\omega_1) = (\mathscr{P}(\omega_1))^{L(\mathbb{R})}$, $\mathcal{L} \restriction A$ is scattered and $\mathcal{L} \restriction A$ is σ-scattered is absolute between the real world and $L(\mathbb{R})$.

The linear ordering $\mathcal{L}$ is still a non-σ-scattered linear ordering in $L(\mathbb{R})$. Kechris showed that if AD holds, then $L(\mathbb{R}) \models \mathsf{AD} + \mathsf{DC}$.

Every linear ordering on ω_1 is σ-Scattered

Theorem
Assume AD*. Every linear ordering on ω_1 is σ-scattered.*

So work inside $L(\mathbb{R})$: Let $\mathcal{I}$ be the σ-ideal of $A \subseteq \omega_1$ so that $\mathcal{L} \restriction A$ is σ-scattered. $\mathcal{I} \neq \mathscr{P}(\omega_1)$ since $\mathcal{L}$ was assumed to be non-σ-scattered. Let $\mathcal{F}$ be the dual filter to $\mathcal{I}$ which is a countably complete ultrafilter.

Kunen showed that any countably complete filter on ω_1 can be extended to an ultrafilter (which is necessarily countably complete): Recall the surjection $\pi : \mathbb{R} \to \mathscr{P}(\omega_1)$ obtained by the coding lemma. Modify π to a surjection $\tilde{\pi} : \mathbb{R} \to \mathcal{F}$. Let μ_{Turing} be the Martin measure on the Turing degrees $\mathcal{D}_{\mathrm{Turing}}$. Let $\Phi : \mathcal{D}_{\mathrm{Turing}} \to \omega_1$ be defined by $\Phi(X) = \min(\bigcap\{\tilde{\pi}(r) : [r]_{\equiv_{\mathrm{Turing}}} \leq_{\mathrm{Turing}} X\})$. Let $\mu = \Phi_* \mu_{\mathrm{Turing}}$ is an ultrafilter extending $\mathcal{F}$.

Since DC holds in $L(\mathbb{R})$, one can use the earlier result which asserts that there is an $A \in \mu$ so that $\mathcal{L} \restriction A$ is scattered. Thus $A \in \mathcal{I}$. So $\omega_1 \setminus A \in \mathcal{F} \subseteq \mu$. So A and $\omega_1 \setminus A \in \mu$ which is a contradiction. This completes the proof.

Almost Everywhere Behavior According to Partition Measure

As promised, we still need to show:

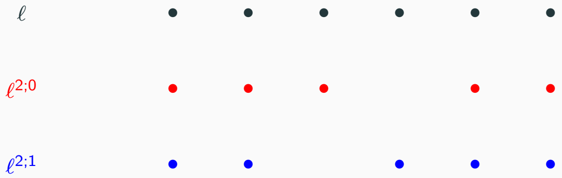
Theorem
Assume $\omega_1 \rightarrow_* (\omega_1)^{n+2}_2$. *Let* $1 \leq n < \omega$ *and* $\mathcal{L} = ([\omega_1]^n, \prec)$ *is a linear ordering on* $[\omega_1]^n$. *There is a club* $C \subseteq \omega_1$, $\tau \in {}^n2$, *and* $\rho \in \mathrm{Bij}_n$ *so that* $\mathcal{L} \restriction [C]^n = \mathcal{L}^{n,\tau,\rho} \restriction [C]^n$.

Fix $1 \leq n < \omega$. For $k < n$, let $T^k = \{(m, 0) : m < n \wedge m \neq k\} \cup \{(k, 0), (k, 1)\}$. Note $|T_k| = n + 1$. Let $\mathcal{T}_k = (T_k, \sqsubset_k)$ where $\sqsubset_k$ is the lexicographic ordering. For $\ell \in [\omega_1]^{\mathcal{T}_k}$ and $i \in 2$, let $\ell^{k;i} \in [\omega_1]^n$ be defined by

$$\ell^{k;i}(m) = \begin{cases} \ell(m,0) & m \neq k \\ \ell(k,i) & m = k \end{cases}.$$

Note that $\ell^{k;0}(k) < \ell^{k;1}(k)$ and for all $m \neq k$, $\ell^{k;0}(m) = \ell^{k,1}(m)$.

For $n = 5$ and $k = 2$, the picture is as follows:



Almost Everywhere Behavior According to Partition Measure

Define $P_k : [\omega_1]^{\mathcal{T}_k} \to 2$ by $P_k(\ell) = 0$ if and only if $\ell^{k;0} \prec \ell^{k;1}$. By $\omega_1 \to_* (\omega_1)^{n+1}_2$, there is a club homogeneous for P_k. Let $\tau(k)$ denote the unique homogeneous value for P_k. This roughly determines the direction of k^{th}-coordinate.

This defines $\tau \in {}^n 2$.

Let $k_0, k_1 \in n$ with $k_0 \neq k_1$. Let $U^{k_0,k_1} = \{(m,0) : m \in n \wedge m \neq k_0 \wedge m \neq k_1\} \cup \{(k_0,0),(k_0,1),(k_1,0),(k_1,1)\}$. Let $\mathcal{U}^{k_0,k_1} = (U^{k_0,k_1}, \sqsubset^{k_0,k_1})$. Let $\iota \in [\omega_1]^{\mathcal{U}^{k_0,k_1}}$. For $i \in 2$, define $\iota^{k_0,k_1;i} \in [\omega_1]^n$ as be defined by

$$\iota^{k_0,k_1;0}(m) = \begin{cases} \iota(m,0) & m \notin \{k_0,k_1\} \\ \iota(m,0) & m = k_0 \\ \iota(m,0) & m = k_1 \wedge \tau(k_0) \neq \tau(k_1) \\ \iota(m,1) & m = k_1 \wedge \tau(k_0) = \tau(k_1) \end{cases}$$

$$\iota^{k_0,k_1;1}(m) = \begin{cases} \iota(m,0) & m \notin \{k_0,k_1\} \\ \iota(m,1) & m = k_0 \\ \iota(m,1) & m = k_1 \wedge \tau(k_0) \neq \tau(k_1) \\ \iota(m,0) & m = k_1 \wedge \tau(k_0) = \tau(k_1) \end{cases}.$$

Define $Q^{k_0,k_1} : [\omega_1]^{\mathcal{U}_{k_0,k_1}} \to 2$ by $Q^{k_0,k_1}(\iota) = 0$ if and only if $\iota^{k_0,k_1;0} \prec \iota^{k_0,k_1;1}$. By $\omega_1 \to_* (\omega_1)^{n+2}_2$, let i^{k_0,k_1} be its unique homogeneous value.

Intuitively, $\iota^{k_0,k_1;0}$ and $\iota^{k_0,k_1;1}$ are mismatching the behavior at the k_0 and k_1 coordinate specified by $\tau(k_0)$ and $\tau(k_1)$ to determine which coordinate is stronger. $i^{k_0,k_1} = 0$ implies that k_0 is stronger than k_1.

Define $\ll$ on n by $k_0 \ll k_1$ if and only if $k_0 \neq k_1$ and $i^{k_0,k_1} = 0$.

With additional combinatorial arguments, one can show that $\ll$ is a linear ordering on n. Let $\rho : n \to n$ be the ranking function corresponding to $\ll$.

With further partition arguments, one will find a club C and argue that $\mathcal{L} \upharpoonright [C]^n = \mathcal{L}^{n,\tau,\rho} \upharpoonright [C]^n$. This sketches the proof of the following:

Theorem
Assume $\omega_1 \to_* (\omega_1)^{n+2}_2$. Let $1 \leq n < \omega$ and $\mathcal{L} = ([\omega_1]^n, \prec)$ is a linear ordering on $[\omega_1]^n$. There is a club $C \subseteq \omega_1$, $\tau \in {}^n2$, and $\rho \in \mathrm{Bij}_n$ so that $\mathcal{L} \upharpoonright [C]^n = \mathcal{L}^{n,\tau,\rho} \upharpoonright [C]^n$.

This was the last remaining ingredient to the proof that every linear ordering on ω_1 is σ-scattered.

Beyond ω_1

One would like to show every linear ordering on ω_2 is σ-scattered. Although ω_2 satisfies $\omega_2 \to_* (\omega_2)^n_2$ for all $n \in \omega$, this is not particularly relevant. Moreover, ω_3 is the first singular cardinal of determinacy and hence does not possess any partition properties.

What is relevant is the measure analysis. Martin showed that for $1 \leq n < \omega$, ω_{n+1} is $\prod_{[\omega_1]^n} \omega_1 / \mu^n_{\omega_1}$. Kunen showed all the measures on ω_{n+1} are Rudin-Keisler equivalent to certain nice measures induced by the strong partition property on ω_1 involving types and uniform cofinalities.

By imitating the argument above but on ω_1-many blocks simultaneously (with the precise arrangement depending on which canonical measure on ω_2 is being analyzed), one should be able to establish an analogous biembeddability basis of scattered linear ordering. Then one should be able to show that for every linear ordering $\mathcal{L} = (\omega_2, \prec)$ and measure μ on ω_2, $\mathcal{L}$ is μ-everywhere scattered. The filter extension argument should complete the argument.

Beyond ω_1

For ordinals below the supremum of the projective ordinals (and even a bit beyond), Jackson classified all the measures on these ordinals. The techniques described above should be able to show that every linear ordering on an ordinal from a small initial segment of Θ are σ-scattered. However Jackson analysis does not extend all the way to Θ.

Inner model theory provides the most powerful technique for proving result in their greatest generality in $L(\mathbb{R})$ or more generally under AD^+. Every linear ordering $\mathcal{L}$ in $L(\mathbb{R})$ on an ordinal belongs to $\mathrm{HOD}_{\{z\}}$ for some $z \in \mathbb{R}$. Every linear orderings in $\mathrm{HOD}_{\{z\}}$ has its origin as a linear ordering in a countable iterable mouse. The intuition is that this countable mouse is the reason why $\mathcal{L}$ is σ-scattered.

We will sketch another proof that every linear ordering on ω_1 is σ-scattered using the HOD-analysis of Steel. This argument is more suitable to generalization to show all linear ordering on an ordinal below Θ is σ-scattered.

Fact (Rowbottom)
If μ is a κ-complete nonprincipal normal ultrafilter on κ, then let μ^n be the measure on $[\kappa]^n$ defined by $B \in \mu^n$ if and only if there exists $A \in \mu$ so that $[A]^n \subseteq B$. μ^n is a κ-complete ultrafilter.

In fact, for every $P : [\kappa]^n \to 2$, there exists an $A \in \mu$ and $i \in 2$ so that $P(\ell) = i$ for all $i \in [A]^n$.

For $\tau \in {}^n2$ and $\rho \in \text{Bij}_n$, one can defined the analogous scattered linear ordering $\mathcal{L}^{n,\tau,\rho}$ on $[\kappa]^n$. The main combinatorial argument above uses the partition property and large homogeneous set according to the partition filter. The same proof establishes the following analog for normal measures.

Theorem
Suppose μ is a normal κ-complete nonprincipal ultrafilter on κ. Let $1 \leq n < \omega$. Suppose $\mathcal{L} = ([\kappa]^n, \prec)$ is a linear ordering. Then there is an $A \in \mu$, $\tau \in {}^n2$, and $\rho \in \text{Bij}_n$ so that $\mathcal{L} \restriction [A]^n = \mathcal{L}^{n,\tau,\rho} \restriction [A]^n$.

Inner Model Theoretic Proof

The argument will be given for $L(\mathbb{R})$ and it can be adapted to AD^+.

Theorem
Assume AD *and* $V = L(\mathbb{R})$. *Every linear ordering on* ω_1 *is* σ*-scattered.*

Let $\mathcal{L} = (\omega_1, \prec)$ be a linear ordering on ω_1. Without loss of generality, suppose $\mathcal{L}$ is OD and hence $\mathcal{L} \in \mathrm{HOD}$.

There is an internal directed system $\mathcal{F} \in L(\mathbb{R})$ which consists of certain countable iterable mice and iteration maps between members of $\mathcal{F}$. If M_∞ denotes the directed limit, then Steel showed $M_\infty = \mathrm{HOD} \cap V_{\delta^2_1}$. Fix some $\mathcal{N}_0 \in \mathcal{F}$ such that there is a $\mathcal{L}_0 = (\delta_0, \prec_0) \in \mathcal{N}_0$ so that $j_{0,\infty}(\mathcal{L}_0) = \mathcal{L}$, where $j_{0,\infty}$ is the directed system map and δ_0 is the least measurable cardinal of $\mathcal{N}_0$.

Note that $j_{0,\infty}(\delta_0) = (\omega_1)^{L(\mathbb{R})}$. Let μ_0 be the unique Mitchell order zero normal measure on δ_0. For each $\alpha < \omega_1$, let N_α be the α^{th}-linear iterate of $\mathcal{N}_0$ by μ_0 and its image. Note that $N_\alpha \in \mathcal{F}$ and the direct system map $j_{\alpha,\beta}$ are the linear iteration maps. Let $\delta_\alpha = j_{0,\alpha}(\delta_0)$ and $\mathcal{L}_\alpha = j_{0,\alpha}(\mathcal{L}_0)$ which takes the form $(\delta_\alpha, \prec_\alpha)$.

Inner Model Theoretic Proof

Let $f : [\delta_0]^n \to \delta_0$ be an element of $\mathcal{N}_0$. Let $\mathcal{J}_0 = ([\delta_0]^n, \sqsubset_0)$ be defined by $\ell_0 \sqsubset_0 \ell_1$ if and only if $f(\ell_0) \prec_0 f(\ell_1)$. Let $\mathcal{J}_\alpha = ([\delta_\alpha]^n, \sqsubset_\alpha)$ be $j_{0,\alpha}(\mathcal{J}_0)$.

Theorem
Suppose μ is a normal κ-complete nonprincipal ultrafilter on κ. Let $1 \leq n < \omega$. Suppose $\mathcal{L} = ([\kappa]^n, \prec)$ is a linear ordering. Then there is an $A \in \mu$, $\tau \in {}^n2$, and $\rho \in \mathrm{Bij}_n$ so that $\mathcal{L} \restriction [A]^n = \mathcal{L}^{n,\tau,\rho} \restriction [A]^n$.

Applying the above Theorem inside of $\mathcal{N}_0$ for $\mathcal{J}_0$, one has that there is some $\tau \in {}^n2$, $\rho \in \mathrm{Bij}_n$, and $A_0 \in \mu_0$ so that $\mathcal{J}_0 \restriction [A_0]^n = \mathcal{L}^{n,\tau,\rho} \restriction [A_0]^n$.

For all $\alpha < (\omega_1)^V$, $\delta_\alpha = j_{0,\alpha}(\delta_0) \in j_{0,\alpha}(A_0)$. Suppose $\ell_0 = (\delta_{\alpha_0}, ..., \delta_{\alpha_{n-1}})$ and $\ell_1 = (\delta_{\beta_0}, ..., \delta_{\beta_{n-1}})$. Let γ be such that $\alpha_{n-1}, \beta_{n-1} < \gamma$. Then $\ell_0, \ell_1 \in [j_{0,\gamma}(A_0)]^n$. By elementarity and the fact that $\mathrm{crit}(j_{0,\gamma}) = \delta_\gamma$, one has that $\ell_0 \sqsubset_\gamma \ell_1$ if and only if $\ell_0 \prec^{n,\tau,\rho} \ell_1$. By applying $j_{\gamma,\infty}$, one has $\ell_0 \sqsubset_\infty \ell_1$ if and only if $\ell_0 \prec^{n,\tau,\rho} \ell_1$.

By the definition of $\mathcal{J}_\alpha$, one has that $j_{0,\gamma}(f)(\delta_{\alpha_0}, ..., \delta_{\alpha_{n-1}}) \prec j_{0,\gamma}(f)(\delta_{\beta_1}, ..., \delta_{\beta_{n-1}})$ if and only if $j_{0,\gamma}(f)(\delta_{\alpha_0}, ..., \delta_{\alpha_{n-1}}) \prec^{n,\tau,\rho} j_{0,\gamma}(f)(\delta_{\beta_1}, ..., \delta_{\beta_{n-1}})$.

Let $E_f = \{j_{0,\gamma}(f)(\delta_{\alpha_0}, ...\delta_{\alpha_{n-1}}) : \alpha_0 < ... < \alpha_{n-1} < \gamma < \omega_1\}$. This shows that $\mathcal{L} \restriction E_f$ is scattered.

For any γ and $x \in \mathcal{L}_\gamma$, x takes the form $j_{0,\gamma}(f)(\delta_{\alpha_0}, ..., \delta_{\alpha_{n-1}})$ for some $n \in \omega$, $\alpha_0 < ... < \alpha_{n-1} < \gamma$, and $f : [\delta_0]^n \to \delta_0$ with $f \in \mathcal{N}_0$. Thus every element of ω_1 takes the form $j_{0,\gamma}(f)(\delta_{\alpha_0}, ..., \delta_{\alpha_{n-1}})$ for some $n \in \omega$, $\alpha_0 < ... < \alpha_{n-1} < \gamma < \omega_1$, and $f : [\delta_0]^n \to \delta_0$ with $f \in \mathcal{N}_0$.

Hence $\bigcup\{E_f : n \in \omega \wedge f : [\delta_0]^n \to \delta_0 \wedge f \in \mathcal{N}_0\} = \omega_1$ and witnesses that $\mathcal{L}$ is σ-scattered since $\mathcal{N}_0$ is countable. This completes the proof.

Theorem
Assume AD *and* $V = L(\mathbb{R})$. *Every linear ordering on* ω_1 *is* σ*-scattered.*

For an arbitrary cardinal below ω_1, one can not longer use linear iterations. Schlutzenberg full normalization results implies the $\mathrm{HOD} \upharpoonright \Theta$ is the iterate of M_ω by a single normal tree. An analysis of the models and embedding along the main branch should serve as an analog of the argument above.

If this works out, then one will show there are no Baumgartner lines on an image of $\mathbb{R}$ and prove the wellorderable Fraïssé conjecture.

Thanks for listening!