Complexity of decision problems in Borel combinatorics and gadget reductions

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Part 1: Setup and survey of related results

Setup: Borel (structured hyper-)graphs

Definition

Let X be a standard Borel space. A Borel graph is a Borel subset of $X \times X$ that is irreflexive and symmetric.

We'll often restrict to bounded degree, locally finite, or locally countable Borel graphs.

We will also consider Borel "structured graphs", which are Borel graphs equipped with a Borel function defined on the vertices and edges. And we will sometimes consider Borel hypergraphs.

Setup: Borel combinatorics

We study classical combinatorial problems on (structured hyper-)graphs but now require the solutions to be Borel functions/sets. Examples are coloring, matching, and orientation problems.

Theorem (Kechris-Solecki-Todorčević, 90's)

A Borel graph with maximum degree d has a proper Borel $(d+1)$-coloring.

Theorem

The Schreier graph of a free Borel action of $\mathbb{Z}^d$ has a proper Borel 3-coloring.

Theorem (Marks, 2013)

The Bernoulli shift $\mathbb{F}_k \curvearrowright \mathbb{N}^{\mathbb{F}_k}$ has no proper Borel $(2k)$-coloring.

So the classical and Borel chromatic numbers can differ, maybe by a lot.

Decision problems in Borel combinatorics

Question

For a fixed "combinatorial problem" $\mathcal{P}$, is there a "simple" characterization of when a given Borel (structured hyper-)graph admits a Borel solution to $\mathcal{P}$?

More precisely, what is the (projective) complexity of

$$\{c \in \mathbb{N}^{\mathbb{N}} : c \text{ is the code of a Borel graph with a Borel solution to } \mathcal{P}\}.$$

Remark

The set of codes for Borel graphs is $\mathbf{\Pi}^1_1$.

Remark

$\{c \in \mathbb{N}^{\mathbb{N}} : c \text{ is a code for a Borel function } f : X \to Y\}$ *is* $\mathbf{\Pi}^1_1$.

Some $\mathbf{\Pi}^1_1$ problems

Theorem (KST, G_0-dichotomy)

For a Borel graph G, exactly one of the following holds:

1. *G has a countable Borel coloring;*
2. *there is a Borel homomorphism from G_0 to G.*

There is an effective strengthening of the G_0-dichotomy, which in particular implies that a lightface Δ^1_1 graph has a countable lightface Δ^1_1 coloring. Hence

$$\{c \in \mathbb{N}^{\mathbb{N}} : c \text{ codes a Borel graph with a countable Borel coloring}\}$$
$$= \{c \in \mathbb{N}^{\mathbb{N}} : c \text{ codes a Borel graph with a countable } \Delta^1_1(c) \text{ coloring}\},$$

which is $\mathbf{\Pi}^1_1$.

Similarly, by the effective L_0-dichotomy for Borel 2-colorings,

$$\{c \in \mathbb{N}^{\mathbb{N}} : c \text{ codes a Borel graph with a Borel 2-coloring}\}$$

is $\mathbf{\Pi}^1_1$.

Borel 3-colorability is complicated

Theorem (Todorčević-Vidnyánszky, 2021)

$\{c \in \mathbb{N}^{\mathbb{N}} : c$ codes a locally finite Borel graph with a Borel 3-coloring$\}$ is $\mathbf{\Sigma}^1_2$-complete.

Recall that a set $B \subseteq Y$ is $\mathbf{\Sigma}^1_2$-complete iff B is $\mathbf{\Sigma}^1_2$ and for every $\mathbf{\Sigma}^1_2$ set $A \subseteq X$ there is a Borel function $f : X \to Y$ s.t. $[x \in A \text{ iff } f(x) \in B]$.

The above theorem shows that there is no simpler characterization of Borel 3-colorability than "There exist Borel sets Red, Blue, Green partitioning the vertex set s.t. all adjacent vertices x, y are in different color classes."

More on Todorčević-Vidnyánszky

Definition

I'll say a decision problem $\mathcal{P}$ on Borel (structured hyper-)graphs is a $\boldsymbol{\Pi}^1_1$-problem if $\{(c, d) :$
c codes a Borel graph G and d codes a Borel solution to $\mathcal{P}$ on $G\}$ is $\boldsymbol{\Pi}^1_1$.

Theorem (Todorčević-Vidnyánszky, 2021)

Let $\mathcal{P}$ be a $\boldsymbol{\Pi}^1_1$-problem. Suppose G is a Borel graph on $[\mathbb{N}]^{\mathbb{N}}$ with no Borel solution to $\mathcal{P}$, but there is a Borel $\Phi : [\mathbb{N}]^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}} \to \mathbb{N}$ s.t. for all $x \in [\mathbb{N}]^{\mathbb{N}}$, $\Phi(x, \cdot)$ solves $\mathcal{P}$ on $G \restriction \{y \in [\mathbb{N}]^{\mathbb{N}} : x \geq^{\infty} y\}$. Then

$$\{c \in \mathbb{N}^{\mathbb{N}} : c \text{ codes a Borel graph with a Borel solution to } \mathcal{P}\}$$

is $\boldsymbol{\Sigma}^1_2$-complete.

Apply the above theorem when $\mathcal{P}$ is Borel 3-colorability and G is the graph induced by $s : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$, $s(x) = x \setminus \{\min x\}$.

Applications of the Todorčević-Vidnyánszky machinery

Theorem (BCGGRV, 2023)

The set of Borel d-regular acyclic graphs admitting a Borel d-coloring is $\mathbf{\Sigma}^1_2$*-complete;*

Theorem (Frisch-Shinko-Vidnyánszky, 2024)

If there is an increasing union of hyperfinite CBERs that is not hyperfinite, then the set of hyperfinite CBERs is $\mathbf{\Sigma}^1_2$*-complete.*

Theorem (Grebík-Higgins)

The set of locally finite Borel graphs with finite Borel asymptotic dimension is $\mathbf{\Sigma}^1_2$*-complete.*

Part 2: Connections to computational complexity and gadget reductions

Constraint satisfaction problems

Definition

For a fixed relational structure H, the associated constraint satisfaction problem $\mathrm{CSP}(H)$ is the problem: given a structure G (over the same relational language), does there exist a homomorphism from G to H?

Examples: k-colorability, k-SAT, solvability of linear systems.

The CSP dichotomy theorem

Theorem (CSP dichotomy theorem)

Let H be a finite relational structure.

1. *If there is a homomorphism $H^4 \to H$ s.t. for all a, e, r, $f(r, a, r, e) = f(a, r, e, a)$, then $\mathrm{CSP}(H)$ is in* P*;*
2. *If there is no such homomorphism, then $\mathrm{CSP}(H)$ is* NP*-complete.*

Theorem (Thornton, 2022)

If H is a finite relational structure and there is no homomorphism $H^4 \to H$ as above, then $\mathrm{CSP}_B(H)$ is $\mathbf{\Sigma}^1_2$-complete. Assuming $\mathrm{P} \neq \mathrm{NP}$*, this shows all* NP*-complete CSP problems have $\mathbf{\Sigma}^1_2$-complete Borel versions!*

If $\mathrm{CSP}(H)$ is in P, then it's a bit more complicated and a few different possibilities arise for $\mathrm{CSP}_B(H)$.

There are other problems besides CSPs...

In some sense, we can code instances of decision problems $\mathcal{P}$ in Borel combinatorics as instances of some $\mathrm{CSP}_B(H)$.

But just showing that $\mathrm{CSP}_B(H)$ is $\boldsymbol{\Sigma}^1_2$-complete is not the same as showing that $\mathcal{P}$ is $\boldsymbol{\Sigma}^1_2$-complete: the complexity may come from instances of $\mathrm{CSP}_B(H)$ that are not instances of $\mathcal{P}$.

Gadget reductions

Theorem (Thornton, 2022)

The set of Borel graphs with a Borel 3-edge coloring is $\mathbf{\Sigma}^1_2$-complete.

One shows 3-edge colorability for finite graphs is NP-complete using a (polynomial-time) gadget reduction from 3SAT to 3-edge colorability. Thornton adapted this construction to the Borel setting.

Gadget reduction from 3-SAT to 3-edge colorability

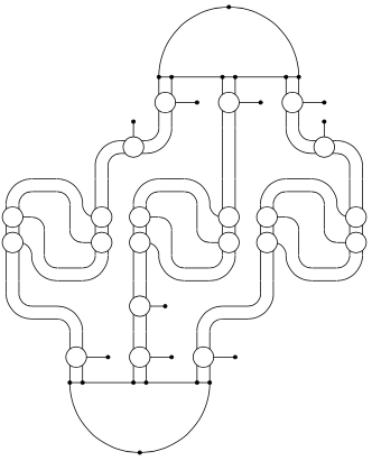


Figure: Reduction applied to $(\neg x_1 \lor x_2 \lor \neg x_3) \land (x_1 \lor \neg x_2 \lor x_3)$

Source: Riley Thornton, "An algebraic approach to Borel CSPs"

Gadget reduction from 3-colorability to 3-colorability for graphs of degree ≤ 5

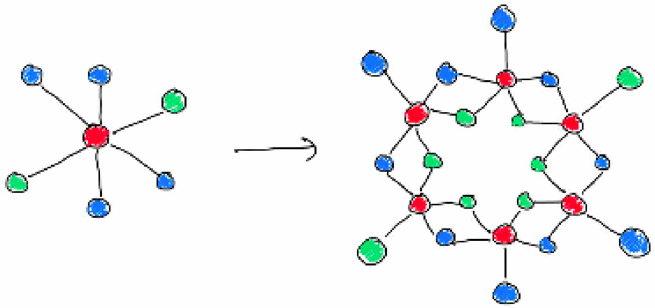


Figure: Replacing a vertex of degree 6 with a gadget of vertices of degree ≤ 5

Gadget reduction from 3-SAT to Hamiltonian path

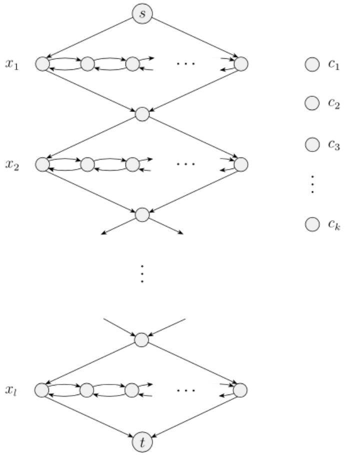


Figure: Reduction applied to 3-SAT formula with ℓ variables and k clauses

Source: Sipser, *Introduction to the theory of computation*

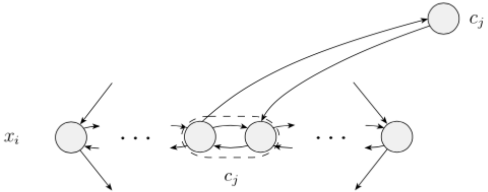


FIGURE 7.51
The additional edges when clause c_j contains x_i

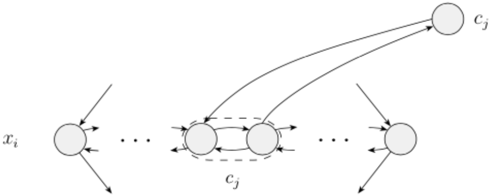
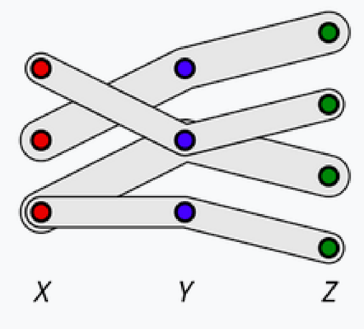
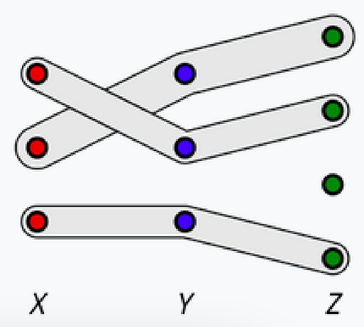


FIGURE 7.52
The additional edges when clause c_j contains $\overline{x_i}$

Other examples of $\mathbf{\Sigma}^1_2$-complete problems using gadget reductions

Borel 3-partite hypergraphs admitting a 3-dimensional perfect matching.





A LOCAL definition of gadget reductions

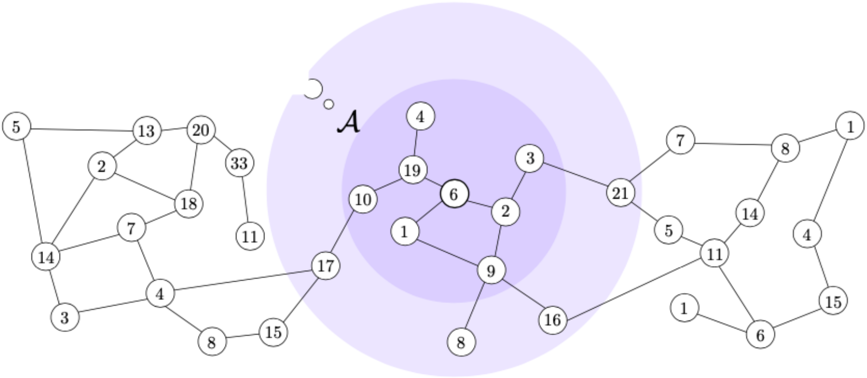


Figure: A deterministic LOCAL algorithm

Source: Václav Rozhoň, "Invitation to Local Algorithms"

A LOCAL definition of gadget reductions

Definition

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be $\mathbf{\Pi}^1_1$ problems on classes of graphs $\mathcal{K}_1$ and $\mathcal{K}_2$. A map $G \mapsto H$ from $\mathcal{K}_1$ to $\mathcal{K}_2$ is a gadget reduction from $\mathcal{P}_1$ to $\mathcal{P}_2$ iff there exists $r > 0$ s.t. for any $G \in \mathcal{K}_1$,

1. for each $v \in V(G)$, there are corresponding vertices in H whose number only depends on $[B_G(v, r)]_{\text{rooted,labeled}}$;
2. for each $v \in V(G)$, there are corresponding edges in H between vertices created by $B_G(v, r)$, and these edges only depend on $[B_G(v, 2r)]_{\text{rooted,labeled}}$;
3. given a $\mathcal{P}_1$-solution to G, we obtain a $\mathcal{P}_2$-solution to H s.t. the $\mathcal{P}_2$-color of each vertex in H created by v only depends on $[B_G(v, r)]_{\text{rooted,labeled},\mathcal{P}_1}$;
4. given a $\mathcal{P}_2$-solution to H, we obtain a $\mathcal{P}_1$-solution to G s.t. the $\mathcal{P}_1$-color of each $v \in V(G)$ only depends on $[B_G(v, 2r)]_{\text{rooted,labeled}}$ and the $\mathcal{P}_2$-colors of the vertices in H created by $B_G(v, r)$.

Gadget reductions from CS imply $\mathbf{\Sigma}^1_2$-completeness results

Theorem

If $\mathcal{P}_1$ is $\mathbf{\Sigma}^1_2$-complete, and there is a gadget reduction from $\mathcal{P}_1$ to $\mathcal{P}_2$, then $\mathcal{P}_2$ is also $\mathbf{\Sigma}^1_2$-complete.

Most NP-complete problems are actually complete for AC^0-reductions, a very strong kind of reduction that seems to "essentially" coincide with the definition of gadget reduction from the previous slide. (I still need to work out the exact relationship.)

Corollary (Informal)

Most NP*-complete $\mathbf{\Pi}^1_1$-problems on (structured hyper-)graphs are $\mathbf{\Sigma}^1_2$-complete.*

Question

Are the Borel versions of all NP*-complete decision problems $\mathbf{\Sigma}^1_2$-complete?*

Gadget reductions from CS imply $\mathbf{\Pi}^1_1$-completeness results

We saw earlier that Borel 2-colorability is $\mathbf{\Pi}^1_1$.

Theorem (Thornton, 2022)

Borel 2SAT *is* $\mathbf{\Pi}^1_1$*.*

It turns out that 2SAT is complete for the computational complexity class NL, even under AC^0-reductions.

Question

Are the Borel versions of all decision problems in NL *of complexity* $\mathbf{\Pi}^1_1$*?*

Other computational complexity classes

If a finitary decision problem is complete for some computational complexity class, then determining the projective complexity of its Borel version has implications for the projective complexities of most other decision problems in that class.

$$\mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE}$$

P-completeness

$\mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE}$

Theorem (Grebík-Vidnyánszky, 2025)

Borel solvability of linear systems over any fixed finite field is $\mathbf{\Sigma}^1_2$-complete.

Question (Should be known)

Is solvability of linear systems P*-complete?*

It turns out Horn-SAT, the problem of deciding if a CNF-formula with at most one negated literal per clause, is P-complete.

Question

What is the projective complexity of Borel Horn-SAT?

Question

Are the Borel versions of all P*-complete decision problems $\mathbf{\Sigma}^1_2$-complete?*

Thanks!