Locally checkable labeling problems in the Borel hierarchy

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Outline

Part I: Introduction

Part II: Warmup: $\texttt{CONTINUOUS}(\mathbb{F}_2) \subsetneq \texttt{BAIRE}_1(\mathbb{F}_2)$

Part III: The inductive step

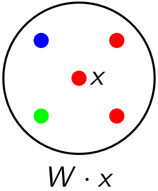
LCLs on groups

Definition

Let Γ be a countable group. An **LCL** on Γ is a triple $\Pi = (\Lambda, W, \mathcal{A})$, where

- Λ is a finite set (**labels**)
- $W \subset \Gamma$ is finite (**window**)
- $\mathcal{A} \subseteq \Lambda^W$ (**allowed configurations**)

- If $\Gamma \curvearrowright X$ is a free action of Γ on a set X, a Π-**labeling** of X (also called a **solution** to Π) is a function $c : X \to \Lambda$ such that for all $x \in X$, the function $W \to \Lambda$ given by $\gamma \mapsto c(\gamma \cdot x)$ is in $\mathcal{A}$.



LCLs on groups

- For example, if $\Gamma = \langle S \rangle$, for $k \in \mathbb{N}$ the problem of (proper) k-coloring the Cayley graph $\text{Cay}(\Gamma, S)$ is an LCL $(k, S \cup \{1\}, \mathcal{A})$, where

$$\mathcal{A} = \{c \mid \forall \gamma \in S, c(1) \neq c(\gamma)\}$$

  That is k-colorings are exactly solutions to this LCL on Γ (with respect to left multiplication).

Some other familiar perspectives:

- Wang tiles (especially for $\Gamma = \mathbb{Z}^2$)
- An SFT $Y \subseteq \Lambda^\Gamma$ is exactly the set of solutions to some LCL on Γ with labels Λ. Solutions to the LCL on X correspond to Γ-equivariant maps $X \to Y$.

Descriptive combinatorics

Given an LCL Π on Γ, we are interested in constructive versions of the question: "does Γ admit a Π labeling?" Versions of this give rise to **complexity classes**.

- Let $\Gamma \curvearrowright X$ be a free Borel action on a standard Borel space. Does X admit a Borel Π-labeling?
  If the answer is always yes, we say $\Pi \in \texttt{BOREL}(\Gamma)$
- Let $\Gamma \curvearrowright (X, \mu)$ be a free Borel action on a standard probability space. Does X admit a μ-measurable Π-labeling?
  If the answer is always yes, we say $\Pi \in \texttt{MEASURE}(\Gamma)$
- Let $\Gamma \curvearrowright X$ be a free continuous action on a **zero dimensional** Polish space. Does X admit a continuous Π-labeling If the answer is always yes, we say $\Pi \in \texttt{CONTINUOUS}(\Gamma)$

Other classes will be defined similarly as we go.

The Bernoulli Shift

- For any Γ, let $\mathcal{S}(\Gamma)$ denote the free part of the Bernoulli shift $\Gamma \curvearrowright (2^\omega)^\Gamma$.
- Any free Borel action $\Gamma \curvearrowright X$ on a standard Borel space admits a Borel equivariant injection to $\mathcal{S}(\Gamma)$. Likewise in the topological setting if X is zero dimensional.
- Thus.

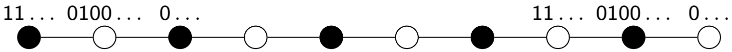
$$\mathtt{BOREL}(\Gamma) = \{\Pi \mid \mathcal{S}(\Gamma) \text{ admits a Borel } \Pi\text{-labeling}\},$$

and likewise for $\mathtt{CONTINUOUS}(\Gamma)$.

Example: 2-coloring

Proposition

$\mathcal{S}(\mathbb{Z})$ does not admit a Borel 2-coloring (with respect to the generating set $\{\pm 1\}$). Thus 2-coloring $\notin$ $\texttt{BOREL}(\mathbb{Z})$.



- We will show that there is no Baire measurable 2-coloring. Suppose there is and let $A \subseteq \mathcal{S}(\mathbb{Z})$ be the set of white points.
- WLOG, A is nonmeager. Let U be a basic open set on which A is comeager. U specifies a finite initial segment of labels of nodes in a finite window around the identity.
- If k is large enough, the windows of U and $k \cdot U$ are disjoint. If k is odd, this leads to a contradiction.

Proposition (Kechris-Solecki-Todorcevic '99)

3-coloring $\in$ $\texttt{CONTINUOUS}(\mathbb{Z})$.

Complexity classes for $\mathbb{Z}$

We have the following picture due to Grebík-Rozhoň ('23)

2-coloring $\in$ Exists a solution on $\mathbb{Z}$

$\uparrow \subsetneq$

3-coloring $\in$ $\texttt{CONTINUOUS}(\mathbb{Z}) = \texttt{BOREL}(\mathbb{Z}) = \texttt{MEASURE}(\mathbb{Z}) = \texttt{BAIREMEAS}(\mathbb{Z}) = \dots$

- $\Pi \in \texttt{BAIREMEAS}(\Gamma)$ if any free Borel action of Γ on a Polish space admits a Baire measurable Π-coloring.

Complexity classes for $\mathbb{F}_n$ ($n \geq 2$)

- We have this picture, combining results of KST ('99), CMT-D ('16), Bernshteyn, BCGGRV ('22), CM ('16), Marks('16).
- All inclusions are strict.
- Note the nontrivial relationship $\texttt{MEASURE}(\mathbb{F}_n) \subsetneq \texttt{BAIREMEAS}(\mathbb{F}_n)$

2-coloring	$\texttt{EXISTS}(\mathbb{F}_n)$
	$\uparrow$
3-coloring	$\texttt{BAIREMEAS}(\mathbb{F}_n)$
	$\uparrow$
$2n$-coloring	$\texttt{MEASURE}(\mathbb{F}_n)$
	$\uparrow$
$2n$-coloring off lines	$\texttt{BOREL}(\mathbb{F}_n)$
	$\uparrow$
$2n+1$-coloring	$\texttt{CONTINUOUS}(\mathbb{F}_n)$

The Baire Hierarchy

Our goal is to understand the interval between $\texttt{CONTINUOUS}(\mathbb{F}_n)$ and $\texttt{BOREL}(\mathbb{F}_n)$.

- Recall that a function $X \to \Lambda$ with X Polish is Baire class α if it is $\mathbf{\Sigma}^0_{1+\alpha}$-measurable.
- Equivalently (since our Λ is finite) $f^{-1}(\lambda) \in \mathbf{\Delta}^0_{1+\alpha}$ for each $\lambda \in \Lambda$.
- For Π an LCL on Γ, say $\Pi \in \texttt{BAIRE}_\alpha(\Gamma)$ if $\mathcal{S}(\Gamma)$ admits a Baire class α Π-labeling. (Equivalently, any zero dimensional Polish Γ-space admits one)
- $\texttt{CONTINUOUS}(\Gamma) = \texttt{BAIRE}_0(\Gamma)$.
- $\texttt{BOREL}(\Gamma) = \bigcup_{\alpha<\omega_1} \texttt{BAIRE}_\alpha(\Gamma)$

Level by level Borel combinatorics

Theorem (Lecomte-Zelený '14, '16)

For each $\alpha < \omega_1$, there is...

- *A $D_2(\mathbf{\Pi}^1_0)$ graph of maximum degree 1 with no Baire class α countable coloring (Any such graph has a Borel 2-coloring).*
- *A closed graph of maximum degree 1 with no Baire class α finite coloring.*

Question (Lecomte-Zelený '14)

Is there a version of the $\mathbb{G}_0$-dichotomy for each level of the Borel hierarchy? That is, a Borel graph G_α such that for each analytic graph H, exactly one:

- *H has a Baire class α countable coloring.*
- *There is a continuous homomorphism $G_\alpha \to H$.*

Level by level Borel combinatorics

Theorem (Marks-Unger '16)

A disc and square of the same area in $\mathbb{R}^2$ are equidecomposable by translations using Borel pieces

- Their pieces are Boolean combinations of $\mathbf{\Sigma}^0_4$ sets, in particular they are $\mathbf{\Delta}^0_5$.

Theorem (Máthé-Noel-Pikhurko '23)

An equidecomposition is possible with pieces that are Boolean combinations of $\mathbf{\Sigma}^0_2$ sets.

Question

What about $\mathbf{\Delta}^0_2$ pieces?

Main Result

Theorem

For each $1 < m \in \mathbb{N}$, *and* $1 \leq n \in \omega$, *there is an LCL* $\Pi = (\Lambda, \dots)$ *on* $\mathbb{F}_m$ *and* $\Lambda_* \subset \Lambda$ *such that*

- $\Pi \in \mathtt{BAIRE}_n(\mathbb{F}_m)$.
- *For any Borel* Π*-labeling* $c : \mathcal{S}(\mathbb{F}_m) \to \Lambda$, $c^{-1}(\Lambda_*)$ *is* $\mathbf{\Pi}^0_n$*-hard. Thus* $\Pi \notin \mathtt{BAIRE}_{n-1}(\mathbb{F}_m)$.

- Note there are only countably many LCLs on a given group!

Question

What is the least α *for which* $\mathtt{BAIRE}_\alpha(\mathbb{F}_m) = \mathtt{BOREL}(\mathbb{F}_m)$.

Main Result

$$\texttt{EXISTS}(\mathbb{F}_n)$$

$$\uparrow$$

$$\texttt{BAIREMEAS}(\mathbb{F}_n)$$

$$\uparrow$$

$$\texttt{MEASURE}(\mathbb{F}_n)$$

$$\uparrow$$

$$\texttt{BOREL}(\mathbb{F}_n)$$

$$\uparrow \; ???$$

$$\texttt{BAIRE}_\omega(\mathbb{F}_n)$$

$$\vdots$$

$$\texttt{BAIRE}_2(\mathbb{F}_n)$$

$$\uparrow$$

$$\texttt{BAIRE}_1(\mathbb{F}_n)$$

$$\uparrow$$

$$\texttt{CONTINUOUS}(\mathbb{F}_n) = \texttt{BAIRE}_0(\mathbb{F}_n)$$

A coloring corollary

Corollary

For any $1 \leq n \in \omega$, there is a connected, locally finite, quasi-transitive graph G such that the Bernoulli shift of G has a Baire class n 3-coloring, but not $n-1$.

- It is easy to see that if such a graph has a Borel 2-coloring, it has a continuous one.

Theorem (Lecomte-Zelený '14, '16)

For each $\alpha < \omega_1$, there is...

- *A $D_2(\mathbf{\Pi}^1_0)$ graph of maximum degree 1 with no Baire class α countable coloring (Any such graph has a Borel 2-coloring).*
- *A closed graph of maximum degree 1 with no Baire class α finite coloring.*

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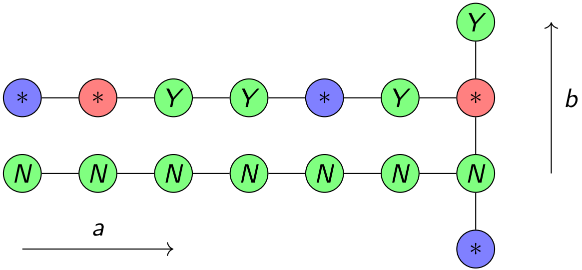
A non local problem

- Since all the $\mathbb{F}_m$'s are subgroups of eachother, it suffices to consider $\mathbb{F}_2 = \langle a, b \rangle$.
- Consider the following (non local) labeling problem with label set $\{Y, N, *\} \times \{R, B, G\}$: First 3-color the b-orbits with $\{R, B, G\}$.
- For the first coordinate, mark all points colored R or B with the $*$. For G points, label them Y if there is some $*$-point in their a-orbit, and N otherwise.

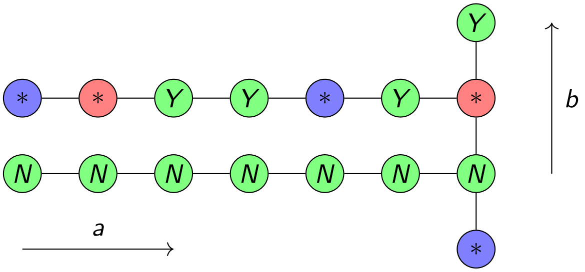


A non local problem

- If $c : \mathcal{S}(\mathbb{F}_2) \to \{R, B, G\}$ is a continuous 3-coloring of the b-orbits, then our resulting $\{Y, N, *\}$ labeling is Baire class 1: Y and N are $\mathbf{\Sigma}^0_1$ and $\mathbf{\Pi}^0_1$ respectively.

Proposition

For **any** *Borel solution $\mathcal{S}(\mathbb{F}_2) \to \{Y, N, *\} \times \{R, B, G\}$, to our problem, the set of points labeled N is $\mathbf{\Pi}^0_1$-hard.*

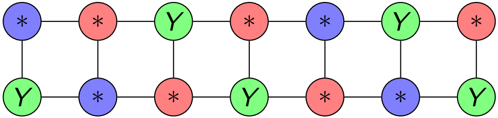


$\mathbb{F}_2$ vs $\mathbb{Z}^2$

- This seems natural, but it is not obvious that the 3-colorings of different b-orbits cannot be correlated in some way.

Theorem (Gao-Jackson-Krohne-Seward)

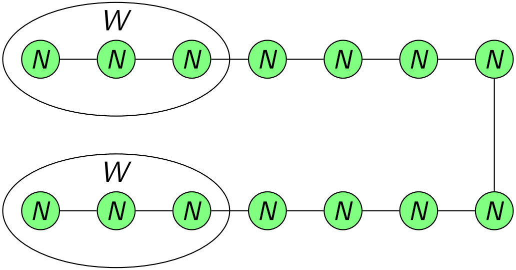
3-coloring $\in \texttt{BOREL}(\mathbb{Z}^2)$.



- Here "N" $= \emptyset$.

N is $\mathbf{\Pi}^0_1$-hard

- We will show $\text{int}(N) = \emptyset$ while $N \neq \emptyset$.
- The former is easy. Suppose $U \subseteq N$ is a basic open, say with window $W \subseteq \mathbb{F}_2$.



N is $\mathbf{\Pi}^0_1$-hard

To show $N \neq \emptyset$, we use:

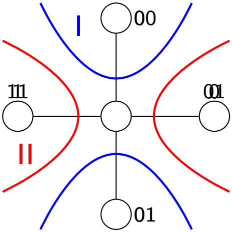
Theorem (Marks '16)

Let Γ, Δ be countable groups, and $A \subseteq \mathcal{S}(\Gamma * \Delta)$ Borel. At least one:

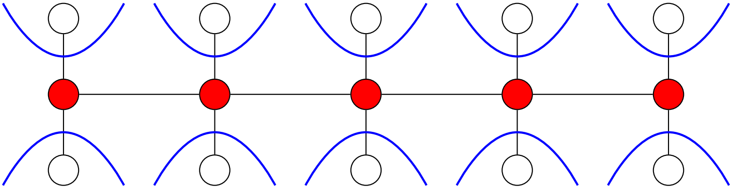
- *There is a continuous Γ-equivariant map $f : \mathcal{S}(\Gamma) \to A$.*
- *There is a continuous Δ-equivariant map $f : \mathcal{S}(\Delta) \to A^c$.*

- We apply this with A the set of G points. In the first case, image$(f) \subseteq N$.
- In the second case, pulling back our coloring along f yields a Borel 2-coloring of $\mathcal{S}(\mathbb{Z})$ (with $\{R, B\}$).

The Marks Game

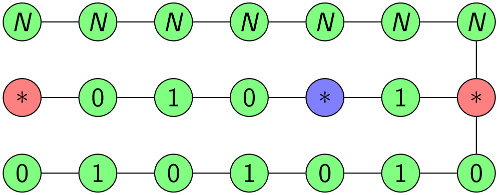


- Consider a game where the players define a point $x \in (2^\omega)^{\Gamma * \Delta}$. I labels "$\Delta$-words" and II Γ. I wins iff $x \in A$.
- Suppose I has a winning strategy. Playing copies of that strategy against eachother gives us an element of A depending in a Γ-equivariant and continuous way on the labels on the red points.

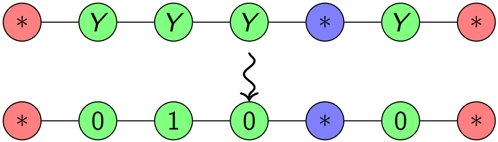


A local problem

- 3-coloring is an LCL. The following additional rules are also local:
  - A point has is labeled with $*$ if and only if it is not G.
  - If x is labeled with N, so are $a \cdot x$ and $a^{-1} \cdot x$.
- But we can't use a local rule to directly force orbits with no $*$ to be labeled N.
- Instead, we use the label set $\{0, 1, N, *\} \times \{R, B, G\}$, and require 0 and 1 to give a partial 2-coloring. This completes our definition of an LCL Π.



A local problem



- To show $\Pi \in \texttt{BAIRE}_1(\mathbb{F}_2)$, we start with a our Baire-1 $\{Y, N, *\} \times \{R, B, G\}$-labeling of $\mathcal{S}(\mathbb{F}_2)$, and note that we can 2-color the Y-points in a relatively continuous way.
- We now want to show that in any Borel Π-labeling of $\mathcal{S}(\mathbb{F}_2)$, the set of points labeled N is $\mathbf{\Pi}^0_1$-hard.
- $\text{int}(N) = \emptyset$ as before, so let us show $N \neq \emptyset$.
- Marks' lemma gave us a continuous a-equivariant map $f : \mathcal{S}(\mathbb{Z}) \to G$. If $N = \emptyset$, the image of this map is Borel 2-colored by $\{0, 1\}$.

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The inductive step

Theorem

Let $n \in \omega$. Γ be a countable group and $\Pi = (\Lambda, \ldots) \in \mathtt{BAIRE}_n(\Gamma)$, and $\Lambda_* \subset \Lambda$ such that for any Borel Π-labeling $c : \mathcal{S}(\Gamma) \to \Lambda$, $c^{-1}(\Lambda_*)$ is $\mathbf{\Pi}^0_n$-hard.
Then there is an $\Pi' = (\Lambda', \ldots) \in \mathtt{BAIRE}_{n+1}(\mathbb{Z} * \Gamma)$, and $\Lambda'_* \subset \Lambda'$ such that for any Borel Π'-labeling $c : \mathcal{S}(\mathbb{Z} * \Gamma) \to \Lambda'$, $c^{-1}(\Lambda'_*)$ is $\mathbf{\Pi}^0_{n+1}$-hard.

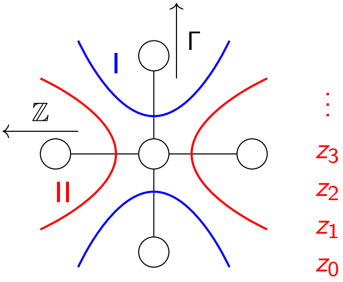
Proposition

Let $c : \mathcal{S}(\mathbb{Z} * \Gamma) \to \Lambda$ be Borel and a Π-labeling of each Γ-orbit. Then

$$N := \{x \mid \mathbb{Z} \cdot x \subseteq c^{-1}(\Lambda \setminus \Lambda_*)\}$$

is $\mathbf{\Pi}^0_{n+1}$-hard.

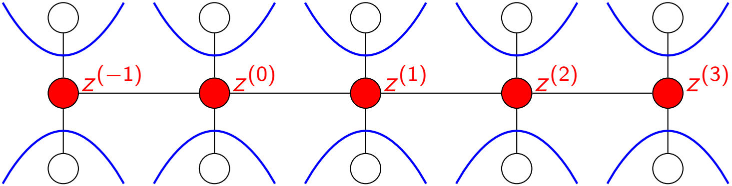
The Marks-Wadge game



- We modify the Marks game so that Player II is also playing a $z \in 2^\omega$ off to the side. Recall the players are constructing some $x \in \mathcal{S}(\mathbb{Z} * \Gamma)$.
- Fix a $\mathbf{\Sigma}^0_n$-complete set $U \subseteq 2^\omega$. II wins if $c(x) \in \Lambda_* \Leftrightarrow z \in U$.

- Suppose II has a winning strategy. We get a continuous Γ-equivariant map $(f, g) : \mathcal{S}(\Gamma) \to \mathcal{S}(\mathbb{Z} * \Gamma) \times (2^\omega)^\Gamma$, the second coordinate recording the z's played by the copies of player II.
- $c \circ f$ is a Borel Π-labeling of $\mathcal{S}(\Gamma)$. g reduces $(c \circ f)^{-1}(\Lambda_*)$ to U, a contradiction.

The Marks-Wadge game



- So I has a winning strategy. We get a $\mathbb{Z}$-equivariant continuous map $f : \mathcal{S}(\mathbb{Z}) \times (2^\omega)^\mathbb{Z} \to \mathcal{S}(\mathbb{Z} * \Gamma)$, the second coordinate giving us z's to feed to player I.
- $c(f(x, (z^{(i)})_{i \in \mathbb{Z}})) \notin \Lambda_* \Leftrightarrow z^{(0)} \in U$.
- So f reduces $U^\mathbb{Z} \subseteq (2^\omega)^\mathbb{Z}$ to $N := \{x \mid \mathbb{Z} \cdot x \subseteq f^{-1}(\Lambda \setminus \Lambda_*)\}$.

Another local problem

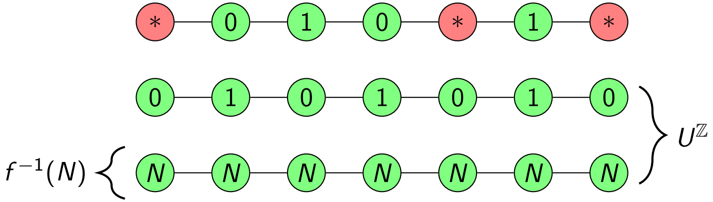
- We turn this into an LCL Π' on $\mathbb{Z} * \Gamma$ with label set $\{0, 1, N, *\} \times \Lambda$ as before:
  - First solve Π on each Γ-orbit.
  - Mark a point with $*$ iff it has a label from Λ_*.
  - If a point is marked with N, so are both its $\mathbb{Z}$-neighbors.
  - The remaining points must be 2-colored by $\{0, 1\}$
- As before, starting with a Baire class n solution to Π gives a Baire class $n + 1$ solution to Π'.

Proposition

Let $c : \mathcal{S}(\mathbb{Z} * \Gamma) \to \{0, 1, N, *\} \times \Lambda$ be a Borel Π'-labeling. Then $c^{-1}(\{N\} \times \Lambda)$ is $\mathbf{\Pi}^0_{n+1}$-hard.

2-coloring and complexity

- We still have a continuous equivariant map $f : \mathcal{S}(\mathbb{Z}) \to \mathcal{S}(\mathbb{Z} * \Gamma)$ so that $f(z)$ is marked with a $*$ if and only if $z^{(0)} \notin U$.
- The issue is that some orbits in $U^{\mathbb{Z}}$ could be entirely 2-colored.



Lemma

Let $U \subseteq 2^\omega$ be $\mathbf{\Sigma}^0_n$-complete. Let $c : U^{\mathbb{Z}} \cap \mathcal{S}(\mathbb{Z}) \rightharpoonup 2$ be a Borel partial 2-coloring with $\mathbb{Z}$-invariant domain. Then $(\mathcal{S}(\mathbb{Z}) \cap U^{\mathbb{Z}}) \setminus \mathrm{dom}(c)$ is $\mathbf{\Pi}^0_{n+1}$-hard (as a subset of $\mathcal{S}(\mathbb{Z})$).

The case $n = 1$:

Lemma

Let $U \subseteq 2^\omega$ be $\mathbf{\Sigma}^0_1$-complete. Let $c : U^\mathbb{Z} \cap \mathcal{S}(\mathbb{Z}) \rightharpoonup 2$ be a Borel partial 2-coloring with $\mathbb{Z}$-invariant domain. Then $(\mathcal{S}(\mathbb{Z}) \cap U^\mathbb{Z}) \setminus \mathrm{dom}(c)$ is $\mathbf{\Pi}^0_2$-hard (as a subset of $\mathcal{S}(\mathbb{Z})$).

- We may assume U is open dense. Then $\mathcal{S}(\mathbb{Z}) \cap U^\mathbb{Z}$ is comeager.
- $\mathrm{dom}(c)$ is meager as we have seen.
- On the other hand, $\mathcal{S}(\mathbb{Z}) \setminus U^\mathbb{Z}$ is dense.

Lemma

If $S \subseteq 2^\omega$ is meager and dense, it is $\mathbf{\Sigma}^0_2$-hard.

Generalizing the lemma

Lemma

If $S \subseteq 2^\omega$ is meager and dense, it is $\mathbf{\Sigma}^0_2$-hard.

Theorem (Day-Marks)

Let $1 \leq n \in \omega$, (X, τ) a Polish space, and $S \subseteq X$. Suppose there is a **suitable sequence of topologies**, $\tau = \tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_n$ *such that*

- *S is τ_n-meager.*
- *For all basic open $W \in \tau_n$, $S \cap W$ is $\tau_{n-1} \restriction W$-comeager.*

Then S is $\mathbf{\Sigma}^0_{n+2}$-hard.

Generalizing Hurewicz's theorem

Localized versions of these criteria provide exact characterizations.

Theorem (Hurewicz)

Let (X, τ) be a Polish space and $S \subseteq X$. S is $\mathbf{\Sigma}^0_2$-hard if and only if there is some closed set $F \subseteq X$ so that, in $\tau \upharpoonright F$, $S \cap F$ is meager and dense.

Theorem (Day-Marks)

Let $1 \leq n \in \omega$, (X, τ) a Polish space, and $S \subseteq X$. S is $\mathbf{\Sigma}^0_{n+2}$-hard if and only if there is some closed set $F \subseteq X$ and a **suitable sequence of topologies**, *$\tau \upharpoonright F = \tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_n$ such that*

- *$S \cap F$ is τ_n-meager.*
- *For all basic open $W \in \tau_n$, $S \cap W$ is $\tau_{n-1} \upharpoonright W$-comeager.*

Applying the Day-Marks criterion

Lemma

Let $U \subseteq 2^\omega$ be $\mathbf{\Sigma}^0_n$-complete. Let $c : U^{\mathbb{Z}} \cap \mathcal{S}(\mathbb{Z}) \rightharpoonup 2$ be a Borel partial 2-coloring with $\mathbb{Z}$-invariant domain. Then $(\mathcal{S}(\mathbb{Z}) \cap U^{\mathbb{Z}}) \setminus \mathrm{dom}(c)$ is $\mathbf{\Pi}^0_{n+1}$-hard (as a subset of $\mathcal{S}(\mathbb{Z})$).

- For an appropriate $\mathbf{\Sigma}^0_n$-complete U, it is not hard to find a suitable sequence $\tau_0 \subseteq \cdots \subseteq \tau_{n-1}$ with τ_0 the usual topology, U open dense in τ_{n-1}, and U F_σ-meager in τ_{n-2}.
- Let $\tau_0^{\mathbb{Z}} \subseteq \cdots \subseteq \tau_{n-1}^{\mathbb{Z}}$ be the product topologies. This turns out to still be suitable.
- $\mathrm{dom}(c)$ will be $\tau_{n-1}^{\mathbb{Z}}$-meager as before.
- On the other hand, since basic open sets in $\tau_{n-1}^{\mathbb{Z}}$ only restrict finitely many coordinates, $U^{\mathbb{Z}}$ will still be relatively $\tau_{n-2}^{\mathbb{Z}}$-meager is any of them.

Further questions

Question

What is the least α *for which* $\mathtt{BAIRE}_\alpha(\mathbb{F}_2) = \mathtt{BOREL}(\mathbb{F}_2)$.

Question

(Assuming PD) Is there an LCL on $\mathbb{F}_2$ *with a* $\mathbf{\Delta}^1_2$*-measurable solution on* $\mathcal{S}(\mathbb{F}_2)$ *but no Borel solution? More generally,* $\mathbf{\Delta}^1_{n+1}$ *but not* $\mathbf{\Delta}^1_n$*?*

Question

What does the $\mathtt{BAIRE}_\alpha(\mathbb{Z}^n)$ *hierarchy look like for* $n > 1$*?*

- Gao-Jackson-Krohne-Seward showed 4-coloring $\in \mathtt{CONTINUOUS}(\mathbb{Z}^n)$ and 3-coloring in $\mathtt{BOREL}(\mathbb{Z}^n) \setminus \mathtt{CONTINUOUS}(\mathbb{Z}^n)$.
- Is 3-coloring $\in \mathtt{BAIRE}_1(\mathbb{Z}^n)$?