Canonicity of the Shift

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joint work with B. Bursics and A. Kocsis

Assume that $V(G)$ is endowed with a Borel structure.
$n \in \{1, 2, \ldots, \aleph_0\}$ is equipped with the trivial Borel structure.
Can talk about:

Borel graphs: G is a Borel graph is G is a symmetric Borel a subset of $V(G) \times V(G)$.

Borel chromatic numbers: minimal n for which G has a Borel n-coloring. Notation: $\chi_B(G)$.

Borel homomorphisms: G admits a Borel homomorphism to H, if there is a Borel map $\varphi : V(G) \to V(H)$ such that
$\forall x, x' \in V(G)((x, x') \in G \implies (\varphi(x), \varphi(x')) \in H)$.
Notation: $G \leq_B H$.

The shift graph

Let $[\mathbb{N}]^{\mathbb{N}}$ denote the set of infinite subsets of $\mathbb{N}$.

Definition

Define the map $\mathcal{S}$ by

$$\mathcal{S}(x) = x \setminus \{min\ x\},$$

and let $xG_{\mathcal{S}}y \iff x = \mathcal{S}(y)$ or $\mathcal{S}(x) = y$.

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More generally, if $f : X \to X$ is a Borel function, define $xG_f y \iff x = f(y)$ or $f(x) = y$.

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More generally, if $f : X \to X$ is a Borel function, define $xG_f y \iff x = f(y)$ or $f(x) = y$.

$\chi_B(G_f) \leq \aleph_0$.

Universality

Theorem

For any acyclic G_f we have $G_f \leq_B G_{\mathcal{S}}$.

Universality

Lemma

There is an f invariant Borel set B so that $G_f \restriction B$ admits a Borel 3-coloring and $G_f \restriction X \setminus B$ admits a Borel homomorphism to $G_{\mathcal{S}}$.

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Fix a Borel $\aleph_0$-coloring $c : X \to \mathbb{N}$ and define $d : X \to 2$ by $d(x) = 0$ iff $c(f(x)) < c(x)$. Let B be the collection of $x \in X$ in front of which both $d(x) = 0$ and $d(x) = 1$ is cofinal, that is,

$$B = \{x : \exists^\infty i \exists^\infty j \; d(f^i(x)) = 0 \wedge d(f^j(x)) = 1\}.$$

Universality

3-coloring B.

Let $C = \{x \in B : d(x) = 1 \wedge d(f(x)) = 0\}$.

C is independent and cofinal in front of every $x \in X$.

Universality

Lemma

1. *$G_f \restriction B$ admits a Borel homomorphism to the acyclic part of $G_{\mathcal{S}_3}$, where $\mathcal{S}_3 : 3^{\mathbb{N}} \to 3^{\mathbb{N}}$ is defined by $\mathcal{S}_3(x)(n) = x(n+1)$.*

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For (2), use toast.

Chromatic numbers

Theorem (Kechris-Solecki-Todořcević, 1999)

There exists a Borel graph $\mathbb{G}_0$ *such that for any Borel graph* G *exactly one of the following holds.*

1. $\chi_B(G) \leq \aleph_0$,
2. $\mathbb{G}_0 \leq_B G$.

Chromatic number of the shift

Proposition (KST)

$\chi_B(G_{\mathcal{S}}) = \aleph_0$.

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Theorem (Galvin-Prikry 1973)

For any $k \in \mathbb{N}$ and any Borel coloring $c : [\mathbb{N}]^{\mathbb{N}} \to k$ there exits an $A \in [\mathbb{N}]^{\mathbb{N}}$ such that c is constant on $[A]^{\mathbb{N}}$.

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Proposition (KST, 1,2,3,∞)

Let $C \subseteq [\mathbb{N}]^{\mathbb{N}}$ be Borel and $\mathcal{S}$-invariant. TFAE:

1. $\chi_B(G_{\mathcal{S}} \restriction C) < \aleph_0$,
2. $\chi_B(G_{\mathcal{S}} \restriction C) \leq 3$,
3. *there is a $C' \subseteq C$ Borel such that both $C \setminus C'$ and C' are cofinal in front of every $x \in C$.*

Characterization

Conjecture

Let C be Borel and $\mathcal{S}$-invariant and assume that $\chi_B(G_{\mathcal{S}} \restriction C) = \aleph_0$ then

1. $[A]^{\mathbb{N}} \subset C$ *for some A?*

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   - (Pequignot 2017) No (BQO theory and Marcone 1994).
   - (Todorčević - V 2021) No, in a strong sense.

Complexity

Theorem (Todorčević - V 2021)

The collection of Borel $C \subset [\mathbb{N}]^{\mathbb{N}}$ *for which* $\chi_B(G_{\mathcal{S}} \restriction C) \leq 3$ *is* Σ^1_2*-complete.*

Understanding complexity

Theorem (Σ^1_2-Determinacy)

A Borel coloring problem is Σ^1_2-complete iff there exists a Borel graph G together with a smooth Borel superequivalence relation F of E_G, so that G does not admit a Borel coloring but it does restricted to every F-class.

Understanding complexity

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Definition

Let G be a Borel graph. Let $\chi_{wB}(G)$ the minimal n such that there exists a smooth Borel superequivalence relation F of E_G, so that G admits a Borel n-coloring restricted to every F-class.

Minimality

Theorem/Conjecture. Let $C \subseteq [\mathbb{N}]^{\mathbb{N}}$ be Borel. Then exactly one of the following holds.

1. $\chi_{wB}(G_{\mathcal{S}} \restriction C) \leq 3$,
2. $G_{\mathcal{S}} \leq_B G_{\mathcal{S}} \restriction C$.

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Let $\alpha \in \omega_1$. For a family $\mathcal{F}$ of finite subset of n define $\mathcal{F}_n = \{t : n < t \wedge \{n\} \cup t \in \mathcal{F}\}$ and $A \subseteq \mathbb{N}$ let $A/n = \{m \in A : m > n\}$.

A family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is α-uniform on $A \subseteq \mathbb{N}$ if

- $\alpha = 0$ and $\mathcal{F} = \emptyset$,

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- $\alpha = 0$ and $\mathcal{F} = \emptyset$,
- $\alpha = \beta + 1$ and for every $n \in \mathbb{N}$ the collection $\mathcal{F}_n$ is β uniform on A/n,
- α is a limit and there is an increasing sequence $\alpha_n < \alpha$ such that $\mathcal{F}_n$ is α_n uniform on A/n.

Canonical forms

Theorem (di Prisco-Todorčević 2006)

For every $\varphi : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ *Borel homomorphism there is an* A*, an* $\alpha < \omega_1$*, an* α*-uniform family* $\mathcal{F}$ *on* A *and a mapping* $\psi : \mathcal{F} \to \mathbb{N}$ *such that for all* $B \in [A]^{\mathbb{N}}$ *we have*

$$\varphi(B) = \{\psi(j(\mathcal{S}^i(B))) : i \in \mathbb{N}\},$$

where $j(B)$ *is the initial segment of* B *in* $\mathcal{F}$*.*

Further directions

Can we characterize $\chi_{wB}(G)$?

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Question

Let G be a 3-regular acyclic Borel graph with $\chi_{wB}(G) = 4$. Is it true that Free$(\mathbb{Z}_2^{\star 3} \curvearrowright 2^{\mathbb{Z}_2^{\star 3}})$ admits a Borel homomorphism to G?

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Is it possible to characterize Borel graphs with $\chi_{wB}(G) \geq \aleph_0$?

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Is it true that a hyper-hyperfinite CBER is weakly hyperfinite?

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What happens with the Borel reducibility hierarchy if we weaken reductions this way?

Thank you for your attention!