A dichotomy theorem for order types of orbit equivalence relations on $\mathbb{R}$

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joint with Eric Paul

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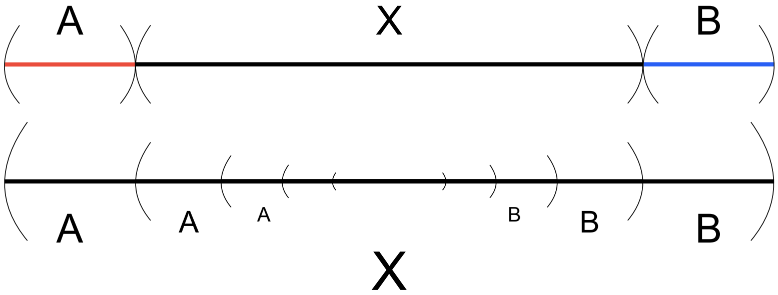
Down the long ladder



Splitting vs. non-splitting orders

A theme: Many structural theorems about linear orders can be viewed as dichotomy theorems that distinguish between:

- Orders that can be split (in some sense) into two separated copies of themselves,
- Orders for which there is no such splitting.

Example: Lindenbaum's splitting theorem

Thm (Lindenbaum-Tarski): Suppose that X is a linear order. Then exactly one holds:

i. $\forall m, n \geq 1$ we have $mX \cong nX$,

ii. $\forall m, n$ such that $m \neq n$ we have $mX \ncong nX$.

Example: Jullien's indecomposability theorem

Thm (Jullien-Hagendorf): Suppose X is an indecomposable linear order. Then exactly one holds:

i. $2X$ embeds in X,

ii. X is strictly indecomposable to the right or left.

Example: Holland's dichotomy theorem

Thm (Holland): Suppose X is a primitive and transitive linear order. Then exactly one holds:

i. X is doubly transitive,
ii. X is uniquely transitive.

Main result

We found a "splitting vs. non-splitting" dichotomy for orbit equivalence relations of subgroups of $\mathrm{Aut}(\mathbb{R}, <)$.

Thm (E., Paul): Suppose G is a subgroup of $\mathrm{Aut}(\mathbb{R}, <)$ whose orbits are each dense in $\mathbb{R}$, and $E = E_G$ is its orbit equivalence relation. Then exactly one holds:

i. $E \cong E_H$ for some group of translations H of $\mathbb{R}$,

ii. $E_B \leqslant E$, where B is the Baumslag-Solitar group $BS(1, 2)$.

??

tail-equivalence

$\mathrm{Aut}(\mathbb{R}, <)$ and its subgroups

- Let Γ denote the group $\mathrm{Aut}(\mathbb{R}, <) = \mathrm{Homeo}_+(\mathbb{R})$.
- For subgroups $G, H \leq \Gamma$, we write $G \cong H$ to mean G is conjugate to H in Γ.
- We view $(\mathbb{R}, +)$ as a subgroup of Γ by identifying $r \in \mathbb{R}$ with the translation $x \mapsto x + r$.
- We say G is a *group of translations* if G is a proper subgroup of $(\mathbb{R}, +)$.

Order-isomorphism of equivalence relations

Def: For E, F equivalence relations on $\mathbb{R}$, we write $E \cong F$ and say E is *order-isomorphic* to F if there is $g \in \Gamma$ such that $gE = F$.

$x E y$ iff $gx F gy$ $\quad \{(gx, gy) : (x,y) \in E\}$

Def: For E, F equivalence relations on $\mathbb{R}$:

i. We write $E \leqslant F$ if there is $g \in \Gamma$ such that $gE \subseteq F$.

ii. We write $E \leqslant_* F$ if there is an order-embedding $g : \mathbb{R} \to \mathbb{R}$ such that $gE \subseteq F$.

strictly incr.

- Observe: if $G \cong H$ then $E_G \cong E_H$.
  - Why: $gGg^{-1} = H$ implies $gE_G = E_H$.
- Converse (very) false in general.

Ordered full groups

Def:

i. For E an equivalence relation on $\mathbb{R}$, define

$$[E] := \{g \in \Gamma : \forall x \in \mathbb{R},\, gx \, E \, x\}.$$

ii. For $G \leq \Gamma$, define

$$[G] := [E_G].$$

We call $[G]$ the *ordered full group* of G.

Easy fact: For $G, H \leq \Gamma$, we have $E_G \cong E_H$ iff $[G] \cong [H]$.

Groups of translations are full

In general, $[G]$ may be much larger than G. But for groups of translations we have the following:

Fact: If G is a group of translations, then $[G] = G$.

Hint: $f \in [G]$

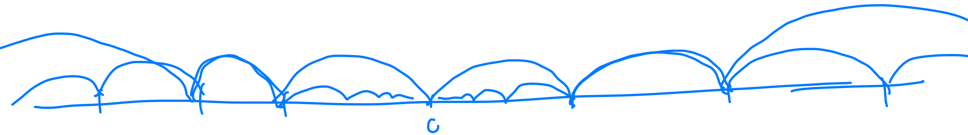
$x \to |f(x) - x|$ is cts

From this and the easy fact from the previous slide, we get:

Fact: If G, H are groups of translations, then $E_G \cong E_H$ if and only if $G \cong H$. More generally, if $G, H \leq \Gamma$ are both conjugate to groups of translations, then $E_G \cong E_H$ if and only if $G \cong H$.

- Groups of translations G, H are conjugate iff one is a scalar multiple of the other.
- This fact combined with the facts above gives that a group of translations G can be recovered up to a scalar factor from the order-type (i.e. $\cong$-type) of E_G.

The Baumslag-Solitar group $BS(1,2)$



- Let $B \leq \Gamma$ denote the group generated by the maps $x \mapsto x + 1$ and $x \mapsto 2x$.
- Then B is the Baumslag-Solitar group $BS(1,2)$.
- We think of its orbit equivalence relation E_B as $\mathbb{R}$'s version of the tail-equivalence relation.

Main result, again

We say $G \leq \Gamma$ is *primitive* if each of its orbits Gx is dense in $\mathbb{R}$.

Here again is our dichotomy theorem:

Thm (E., Paul): Suppose $G \leq \Gamma$ is primitive. Then exactly one holds:

i. $E_G \cong E_H$ for some group of translations H,
ii. $E_B \leqslant_{*} E_G$.

Toward a proof: sums of linear orders

Def: Given linear orders X and Y:

i. The *sum* $X + Y$ is the order obtained by placing a copy of Y to the right of X ("X followed by Y").

$X =$ $\qquad$ $X+Y =$

$Y =$

ii. For $n \in \mathbb{N}$, nX denotes the n-fold sum $X + X + \cdots + X$.

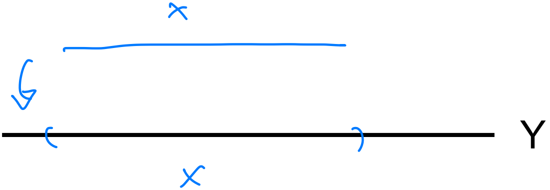
$3X =$

Isomorphisms and convex embeddings

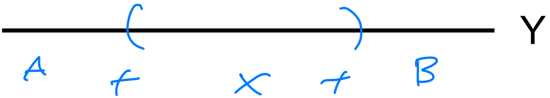
Def: Given linear orders X and Y:

i. Write $X \cong Y$ if X is isomorphic to Y.

ii. Write $X \leq_{conv} Y$ if there is a convex embedding of X in Y.



<u>Observe</u>: $X \leq_{conv} Y$ iff $Y \cong A + X + B$ for some (possibly empty) orders A and B.



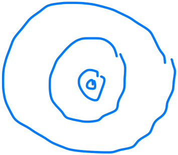
A fundamental observation

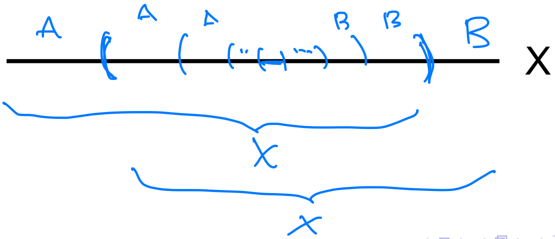
Until further notice let $A, B, X, Y, \ldots$ denote linear orders.

Many structural results concerning automorphisms and convex embeddings of linear orders depend on the following observation.

Prop'n (Lindenbaum): If $X \cong A + X + B$, then $X \cong A + X$ and $X \cong X + B$.

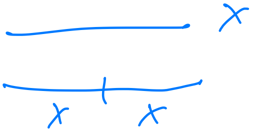
Proof.





□

Splitting lemma



Def: We say X is *splitting* if $2X \cong X$.

<u>Observe</u>: If X is splitting, then $mX \cong nX$ for any $m, n \geq 1$.

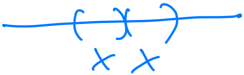
Lem (Splitting Lemma): $2X \cong X$ iff $2X \leq_{conv} X$.

Proof.

$$X = A) + X + (X + B$$

$$X = X + X + B$$

$$X = X + X$$



$\square$

Splitting theorem

Thm (Splitting Theorem) (Lindenbaum): TFAE:

i. $2X \cong X$,

ii. $\forall m, n \geq 1$ we have $mX \cong nX$,

iii. $\exists m \geq 1$ such that $(m+1)X \leq_{conv} mX$.

Proof.

$\square$

Global colorings of linear orders

- We'll use Lindenbaum's results about splitting orders to prove our dichotomy theorem.
- First we generalize these results to linear orders that are colored by a global coloring scheme.

Def: Suppose $\mathcal{C}$ is a class of colors that we use to color every linear order X.

We write $\mathrm{color}(x)$ for the color (from $\mathcal{C}$) of a given point x from a given order X.

Sums, isomorphisms, embeddings of orders with color

Def: Given linear orders X and Y colored by our color scheme:

i. The *sum* $X + Y$ is the order obtained by placing a (colored) copy of Y to the right of X.

$X =$

$Y =$

$X + Y =$

ii. We write $X \cong Y$ if there is an isomorphism $f : X \to Y$ such that $\mathrm{color}(x) = \mathrm{color}(f(x))$ for all $x \in X$.

iii. We write $X \leq_{conv} Y$ if there is a convex embedding $f : X \to Y$ such that $\mathrm{color}(x) = \mathrm{color}(f(x))$ for all $x \in X$.

- Lindenbaum's proposition

$$X \cong A + X + B \Rightarrow (X \cong A + X) \wedge (X \cong X + B)$$

remains true for orders with colorings.

  - Why: the maps we constructed witnessing the isomorphisms on the right were piecewise combinations of the isomorphism from the left and the identity, both of which are color-preserving.

- It follows that the Splitting Lemma and Splitting Theorem remain true for orders with colorings.

Automorphisms of linear orders

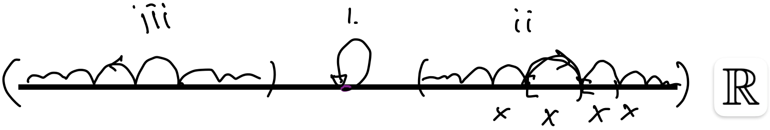
- To see how Lindenbaum's results can help us prove our dichotomy theorem, it will help to understand what order-automorphisms of $\mathbb{R}$ look like.

- Observe: if $f : \mathbb{R} \to \mathbb{R}$ is an element of $\Gamma = \mathrm{Aut}(\mathbb{R}, <)$ and $x \in \mathbb{R}$, then exactly one holds:

  i. $f(x) = x$,

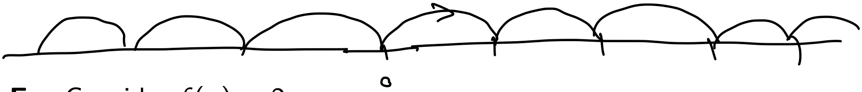
  ii. $\ldots < f^{-1}(x) < x < f(x) < f^2(x) < \ldots$,

  iii. $\ldots < f^2(x) < f(x) < x < f^{-1}(x) < \ldots$.

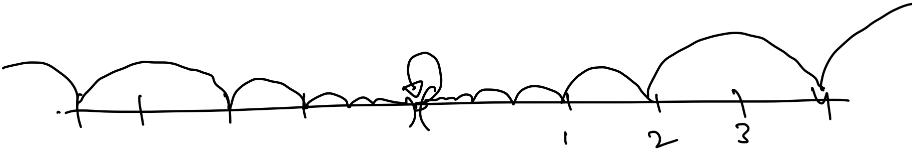


Examples

Ex. Consider $g(x) = x + 1$.



Ex. Consider $f(x) = 2x$.

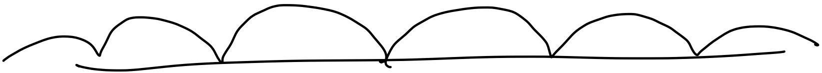


Irreducible automorphisms

Def: Suppose $f \in \Gamma$ and $x \in \mathbb{R}$.

i. The *orbit* of x under f is $o_f(x) := \{f^n(x) : n \in \mathbb{Z}\}$.
ii. The *orbital* of x under f is $O_f(x) :=$ the convex closure of $o_f(x)$.

Def: We say $f \in \Gamma$ is *irreducible* if for some (equiv. any) $x \in \mathbb{R}$, we have $O_f(x) = \mathbb{R}$.

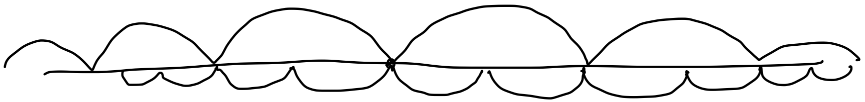


For example, $f(x) = x + 1$ is irreducible (as is any non-identity translation), whereas $g(x) = 2x$ is not.

The Hölder-Conrad Theorem

Hölder proved a theorem, later improved by Conrad, showing that groups $G \leq \Gamma$ consisting only of irreducible automorphisms are essentially groups of translations.

Thm (Hölder-Conrad): Suppose $G \leq \Gamma$ is primitive. If every $g \in G$ is irreducible, then $G \cong H$ for some group of translations H.



- It follows from Hölder-Conrad that to prove our dichotomy theorem, it suffices to show the following:

  Claim: If G is a primitive group of order-automorphisms of $\mathbb{R}$ and there is a non-irreducible $g \in G$, then $E_B \leqslant E_G$.

- We'll sketch in this case (using Lindenbaum's results) that $E_B \leqslant_* E_G$.

B splits its segments

Obs:

- Let $X = [0, 1)$ and view X as being colored by the orbit equivalence relation E_B.
- Then $X \cong 2X$:

$g(x) = x+1$

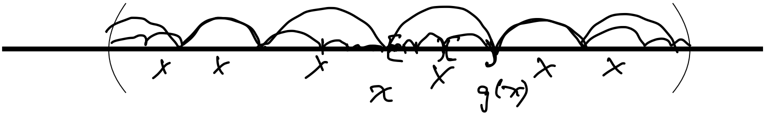
$f(x) = 2x$

If G splits some segment, $E_B \leqslant_* E_G$

Idea:

- Sps $G \leq \Gamma$ and $g \in G$ is increasing on the orbital $O_g(x)$ for some $x \in \mathbb{R}$.
- Let $X = [x, g(x))$ and view X as colored by E_G.
- If $X \cong 2X$, we can find a copy of $B = BS(1,2)$ acting on $O_g(x)$ that preserves E_G:

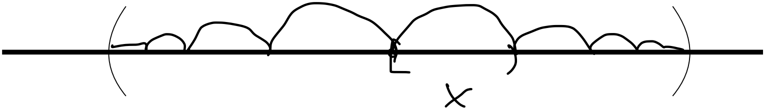
f $\qquad$ $fgf^{-1} = g^2$



- This then gives $E_B \leqslant_* E_G$.

Proof of the dichotomy theorem

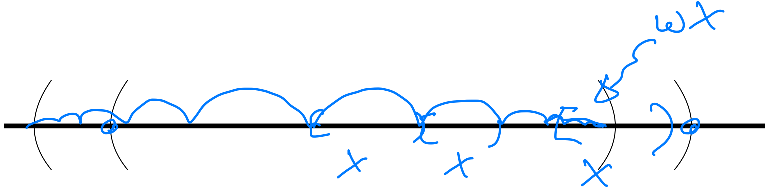
- So suppose $E = E_G$ is the orbit equivalence relation of some primitive $G \leq \Gamma$, and $E \not\cong E_H$ for any group of translations H.
- By Hölder-Conrad, there is $g \in G$ with a bounded (say, on the right) orbital $O_g(x)$:



- Let $X = [x, g(x))$, and view X as being colored by E_G.

Proof of the dichotomy theorem

- By primitivity, we can slide the right endpoint of this orbital slightly to the left.
- Then the right side of the image orbital lies inside the original orbital:



- The ω-tail of X's in the jostled orbital lie in some copy of X in the original.
- In particular $2X \leq_{conv} X$, so by Lindenbaum, $2X \cong X$. $\square$

Turning $\leqslant_*$ to $\leqslant$

- In the argument above, if we could instead find $g \in [G]$ with
  i. $O_g(x) = \mathbb{R}$ and
  ii. $X = [x, g(x)) \cong 2X$,

  such that

  iii. B acts *primitively* on $O_g(x)$,

  then we could conclude $E_B \leqslant E$.
- (*i.*) and (*ii.*) are always possible; (*iii.*) is frequently possible and likely always possible.

Thank you!