# Simplicial complexes, stellar moves, projective amalgamation, and set theory

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**(projective) amalgamation** ⟶ {
- (projective) Fraïssé limits
- (dual) Ramsey theory
- topological dynamics/ extreme amenability

There is an issue of **one-dimensionality** on the right-hand side.

A **simplicial complex** is a family $A$ of non-empty finite sets closed under taking non-empty subsets and such that

$$\mathrm{Vr}(A) \cap A = \emptyset,$$

where $\mathrm{Vr}(A) =$ the union of all sets in $A$.

Sets in $A$ are **faces** of $A$.

Elements of $\mathrm{Vr}(A)$ are **vertices** of $A$.

A **simplicial map** $f\colon A \to B$ is a function $f\colon \mathrm{Vr}(A) \to \mathrm{Vr}(B)$ such that

$$s \in A \Rightarrow f(s) \in B.$$

# Stellar moves and geometric realization

**Alexander**, **Newman**, 1926–1931

stellar moves $\longrightarrow \begin{cases} \text{subdivision} \\ \\ \text{welding} \end{cases}$

$A$ a simplicial complex, $s$ a non-empty finite set

**Subdivision** $sA$ of $A$ by $s$ is defined as follows.

Fix a **new** vertex $s^\vee$.

Declare $sA$ to consist of

$$\begin{cases} y \cup \{s^\vee\}, & \text{if } s \not\subseteq y \text{ and } s \cup y \in A; \\ y, & \text{if } s \not\subseteq y \text{ and } y \in A. \end{cases}$$

The family $sA$ is a simplicial complex.

**Welding** is the **inverse** operation to subdivision.

$$\begin{cases} y \cup \{s^{\vee}\}, & \text{if } s \not\subseteq y \text{ and } s \cup y \in A; \\ y, & \text{if } s \not\subseteq y \text{ and } y \in A. \end{cases}$$

$$\textbf{simplicial complex} \longrightarrow \begin{cases} \text{geometric realization} \\ \\ \text{set theoretic realization} \end{cases}$$

A **geometric realization** is determined by

$$r\colon \mathrm{Vr}(A) \to \mathbb{R}^n$$

such that
for $s \in A$, the points $r(v)$, with $v \in s$, are in **general position** and
for $s, t \in A$,

$$\mathrm{conv}\big(r(s)\big) \cap \mathrm{conv}\big(r(t)\big) = \mathrm{conv}\big(r(s \cap t)\big).$$

The geometric realization is

$$|A|_r = \bigcup_{s \in A} \mathrm{conv}\big(r(s)\big).$$

**Adiprasito–Pak**, 2024

$A$, $B$ simplicial complexes that have

$$r_A \colon \mathrm{Vr}(A) \to \mathbb{R}^n,\ r_B \colon \mathrm{Vr}(B) \to \mathbb{R}^n,$$

determining geometric realizations of $A$ and $B$ such that

$$|A|_{r_A} = |B|_{r_B}.$$

Then there are iterated **subdivisions** $A'$ of $A$ and $B'$ of $B$ such that $A'$ **and** $B'$ **are isomorphic**.

# Weld-division maps

**Aim**: carry over stellar moves from simplicial complexes to simplicial maps:

— **refine** the weld operation to define a class of simplicial maps called **weld maps**

— **lift** the subdivision operation to define an operation on simplicial maps called **subdivision**

## Weld maps

$A$ a simplicial complex, $s$ a finite non-empty set, $x \in s$

The **weld map**

$$\pi^A_{x,s} \colon sA \to A$$

maps each vertex in $\mathrm{Vr}(A)$ to itself,

maps the new vertex $s^\vee$ of $sA$ to $x$, when $s \in A$, that is,

$$s^\vee \to x.$$

$\pi^A_{x,s}$ is a simplicial map.

## Subdivision of simplicial maps

$B$ a simplicial complex, $S \subseteq B$

$S$ is **additive** if, for $s, t \in S$,

$$s \cup t \in B \Rightarrow s \cup t \in S.$$

If $\vec{S}_1$ and $\vec{S}_2$ are non-decreasing (with respect to $\subseteq$) enumerations of $S$, then

$$\vec{S}_1 B = \vec{S}_2 B.$$

We write

$$SB.$$

$f\colon B \to A$ a simplicial map, $s$ be a non-empty finite set

Consider

$$f^{-1}(s) = \{t \in B\colon f(t) = s\}.$$

$f^{-1}(s)$ is an additive family of faces of $B$.

$$sf\colon \bigl(f^{-1}(s)\bigr)B \to sA$$

maps $t^{\vee}$, for each $t \in f^{-1}(s)$, to $s^{\vee}$

maps $v$ of $B$ to $f(v)$.

The map $sf$ is simplicial.

$sf$ is called a **subdivision** of $f$ by $s$.

**Weld-division maps** = simplicial maps obtained from **weld maps** using **subdivision** of simplicial maps and **composition**.

# The category $\mathcal{D}(\mathsf{A})$ and the amalgamation theorem

Fix a simplicial complex $\mathbf{A}$

**Objects** = all simplicial complexes obtained from $\mathbf{A}$ by iterated subdivision (taken up to isomorphisms preserving the face structure)

**Morphisms** = all weld-division maps among above objects

The category above is called the **weld-division category** and is denoted by

$$\mathcal{D}(\mathbf{A}).$$

**Theorem (S.)**

*For $f', g' \in \mathcal{D}(\mathbf{A})$ with the same codomain, there exist $f, g \in \mathcal{D}(\mathbf{A})$ such that*

$$f' \circ f = g' \circ g.$$

So, $\mathcal{D}(\mathbf{A})$ fulfills the **projective amalgamation property**.

# Consequences in projective Fraïssé theory

For a simplicial complex $A$, consider its reduct

$$(\mathrm{Vr}(A), R^A),$$

where

$$aR^Ab \Leftrightarrow a \text{ and } b \text{ belong to a face of } A.$$

$\mathcal{D}_R(\mathbf{A})$ = the category with the objects above, where $A$ is an iterated subdivision of $\mathbf{A}$, and the same morphisms as in $\mathcal{D}(\mathbf{A})$

**Corollary**

*$\mathcal{D}_R(\mathbf{A})$ is a projective Fraïssé class.*

$\mathcal{D}_R(\mathbf{A})$ has a (unique up to an isomorphism) **generic projective sequence**

$$\mathbf{A} = A_0 \xleftarrow{f_0} A_1 \xleftarrow{f_1} A_2 \xleftarrow{f_2} A_3 \xleftarrow{f_3} \cdots$$

with $f_0, f_2, \ldots$ morphisms in $\mathcal{D}_R(\mathbf{A})$, so weld-division maps.

**projective Fraïssé limit** $(\mathbb{A}, R^{\mathbb{A}})$ of $\mathcal{D}_R(\mathbf{A})$ = **inverse limit** of the sequence above

**Theorem (S.)**

(i) *The binary relation $R^{\mathbb{A}}$ is a compact equivalence relation on $\mathbb{A}$.*

(ii) *$\mathbb{A}/R^{\mathbb{A}}$ is homeomorphic to a geometric realization of $\mathbf{A}$.*

# The proof of amalgamation and set theory

## Set theoretic realization of simplicial complexes

$\mathrm{Ur}$ = a set of urelements

$\mathrm{Fin}^+$ = all sets obtained from $\mathrm{Ur}$ by iteratively applying the operation "take all finite non-empty subsets"

The above is after **J. Barwise**, *Admissible Sets*.

**A** a simplicial complex with $\mathrm{Vr}(\mathbf{A}) \subseteq \mathrm{Ur}$

$A \subseteq \mathrm{Fin}^+$ a simplicial complex with $t_1 \notin \mathrm{tc}(t_2)$, for $t_1, t_2 \in A$

$s \in \mathrm{Fin}^+$

Declare $sA$ to consist of

$$\begin{cases} y \cup \{s\}, & \text{if } s \not\subseteq y \text{ and } s \cup y \in A; \\ y, & \text{if } s \not\subseteq y \text{ and } y \in A. \end{cases}$$

$sA \subseteq \mathrm{Fin}^+$ is a simplicial complex with $t_1 \notin \mathrm{tc}(t_2)$, for $t_1, t_2 \in sA$

**Note**: if $s$ is a face of $A$, the new vertex in $sA$ is $s$.

## Formal definition of $\mathcal{D}(\mathbf{A})$

For a sequence of sets $s_0 \cdots s_l$ in $\mathrm{Fin}^+$, the objects are simplicial complexes

$$s_0 \cdots s_l \mathbf{A}$$

obtained as iterated subdivisions of **A**.

**Weld maps** and **subdivision of simplicial maps** are defined by the **same formulas** as before.

We add **combinatorial isomorphisms**.

## Isomorphisms

**Type 1**: $t$ a face of $A$, $r, s \subseteq t$, $r \cup s \neq \emptyset$, $r \cap s = \emptyset$; then

$$t \to s \cup \{t\},\ r \cup \{t\} \to t$$

is an isomorphism

$$\text{from} \quad \big(r \cup \{t\}\big)(r \cup s)\, t\, A \quad \text{to} \quad \big(s \cup \{t\}\big)(r \cup s)\, t\, A.$$

**Type 2**: $s, t$ faces of $A$; then

$$(s \setminus t) \cup \{t\} \to (t \setminus s) \cup \{s\}$$

is an isomorphism

$$\text{from } \big((s \setminus t) \cup \{t\}\big)\, s\, t\, A \;\text{ to } \big((t \setminus s) \cup \{s\}\big)\, t\, s\, A.$$

**Type 3**: $\{x\}$ a face of $A$; then

$$x \leftrightarrow \{x\}$$

are isomorphisms

between $A$ and $\{x\}\,A$.

**Combinatorial isomorphisms** are maps generated by isomorphisms of type 1–3 by composition and subdivision.

# Questions

**1.** Work out a precise framework of the calculus of sequences of finite sets.

**2.** Does projective amalgamation hold for the category generated by welds and combinatorial isomorphisms?

**3.** Are subdivisions of a fixed simplicial complex rigid?

**4.** Does the **dual Ramsey theorem** hold for $\mathcal{D}(\mathbf{A})$ ?