Graphings of Arithmetical Equivalence Relations

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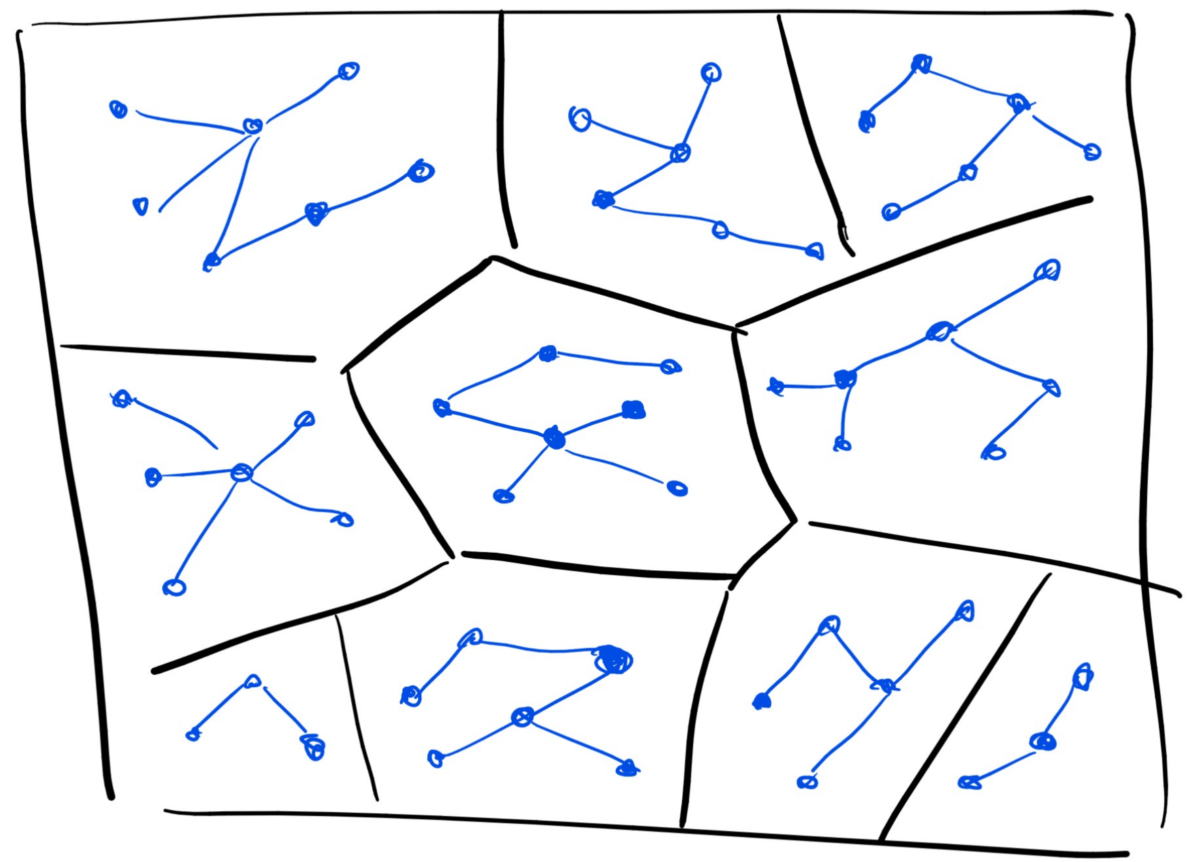


Definitions. 1) A **graphing** of an equivalence relation E is a (simple, undirected) graph G whose connectivity eq. rel. is equal to E, i.e.,

$x E y \iff$ there is a path from x to y in G

2) Γ a pointclass, E eq. rel. on a Polish space X. E is **Γ-graphable** if E has a graphing G which is in Γ.

Arant - Kechris - Lutz : Borel graphable eq. rel.'s

$\to$ Studied analytic eq. rel's which are **Borel** graphable.

A quick "plug":

ω_1^x = least ordinal which has no x-recursive presentation

$$x \, F_{\omega_1} \, y :\iff \omega_1^x = \omega_1^y \qquad \left(\begin{array}{c}\text{eq. rel.}\\ \text{on } 2^{\mathbb{N}}\end{array}\right)$$

Theorem. (AKL) F_{ω_1} is Borel graphable iff there is a nonconstructible real.

More relevant for today:

Theorem. (AKL) $\mathcal{L}$ a countable (first-order) language.
Isomorphism of $\mathcal{L}$-structures (presented on $\mathbb{N}$) is
Borel graphable.

... What about $\underbrace{\text{computable isomorphism}}_{\text{arithmetical eq. rel.}}$?

<u>New work</u>. Given an arithmetical eq. rel. E, does E admit a definable graphing which is simpler than E?

i.e. lower down in arithmetical hierarchy

$\exists n \forall m \exists k \, R(x,n,m,k)$

$$\begin{array}{ccccccc} & \Sigma^0_1 & & \Sigma^0_2 & & \Sigma^0_3 & \\ \Delta^0_1 & & \Delta^0_2 & & \Delta^0_3 & & \cdots \\ & \Pi^0_1 & & \Pi^0_2 & & \Pi^0_3 & \end{array}$$

A familiar example

E_0, eventual equality on $2^{\mathbb{N}}$,

$$x \, E_0 \, y :\Longleftrightarrow (\exists n)(\forall m > n)\left[x(m) = y(m)\right]$$

- E_0 is a countable eq. rel. (i.e., its classes are countable)
- E_0 is $\Sigma^0_2 \setminus \underset{\sim}{\Pi}^0_2$
- E_0 has a "nice" graphing, G_0.

Definition of G_0:

Fix a computable, dense $\{\tau_n\} \subseteq 2^{<\mathbb{N}}$

Edges:

$$\begin{array}{c} \tau_n{}^\frown 0{}^\frown x \\ \big| \; G_0 \\ \tau_n{}^\frown 1{}^\frown x \end{array} \qquad (x \in 2^{\mathbb{N}})$$

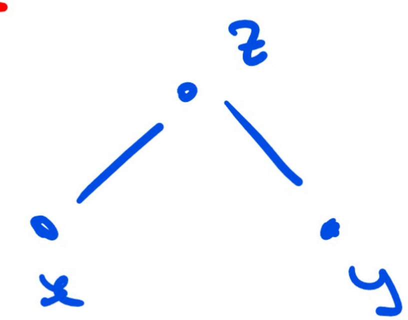
Observation: G_0 is a Δ^0_2 graphing of the Σ^0_2 eq. rel. E_0.

The diameter of the graphing

The diameter of a graph G is the least k s.t. every pair of connected vertices has a path between them of length $\leq k$.

An equivalence relation E is Γ-graphable with diameter k if it has a Γ graphing G whose diameter is k.

In the study of definable graphings of eq. rel's, <u>diameter 2</u> is very common.



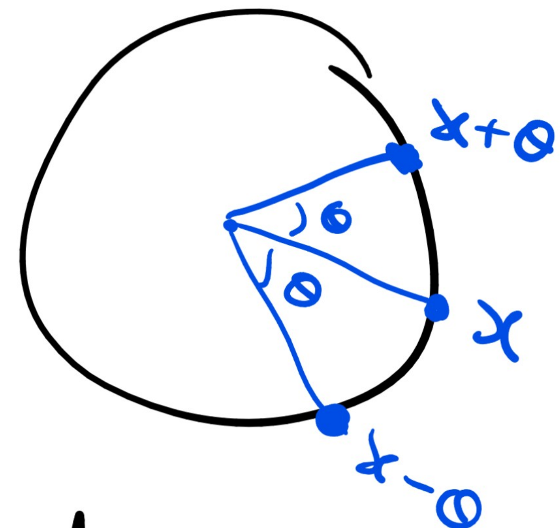
<u>Open</u>: Is there a Borel graphable equivalence relation E s.t. all of its Borel graphings have diameter ≥ 3?

Irrational rotations of the circle

Let E_θ be the orbit eq. rel. of rotation by a computable irrational θ.

E_θ is Π^0_1 graphable,

$$x \, G \, y \iff x \pm \theta = y$$



G does not have finite diameter.

Theorem (A.) E_0 does <u>not</u> have any closed graphings of finite diameter.

<u>Proof</u> Suppose G is a closed graphing of diameter k.

Fix some x, $\mathcal{O}(x)$ orbit of x.

$\uparrow$ dense in the circle

$$R(x_1, \ldots, x_k) \iff x \, G \, x_1 \wedge (\forall i < k)\left[x_i = x_{i+1} \vee x_i \, G \, x_{i+1} \right]$$

R is closed.

Fix $y \notin \Theta(x)$. By denseness,

Pick some (y_n) in $O(x)$, $y_n \longrightarrow y$

Pick $x_{1,n}, \ldots, x_{k-1,n}$ s.t.

$$R(x_{1,n}, \ldots, x_{k-1,n}, y_n)$$

By compactness,

R holds

since closed

$$(x_{1,n}, \ldots, x_{k-1,n}, y_n) \longrightarrow (x_1, \ldots, x_{k-1}, y)$$

$$\Longrightarrow y \in O(x)$$

Theorem (A.) All of the following eq. rel's are Π^0_2-graphable with diameter 2:

- $\equiv_T$ (Turing equivalence) $\leftarrow$ Folklore
- $\equiv_1$, $\equiv_m$ (1-equivalence, m-equivalence of sets)
- Computable isomorphism of $\mathcal{L}$-structures on $\mathbb{N}$, where $\mathcal{L}$ is a computable (first-order) language.

Moreover, all of their Friedman-Stanley jumps (of any finite order) are also Π^0_2-graphable with diameter 2.

Theorem. (Folklore) $\equiv_T$ is Π^0_2 graphable with diameter 2.

1-equivalence

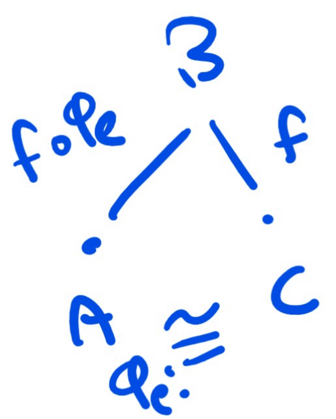
For $A, B \subseteq \mathbb{N}$, define

$A \equiv_1 B \iff$ there is a computable bijection $f: A \to B$ s.t. $n \in A \iff f(n) \in B$

* $\equiv_1$ is Σ^0_3 and, by a result of Rossegger-Slaman-Steifer, it is <u>not</u> Π^0_3.

Proof sketch for $\equiv_1$ being Π^0_2-graphable:

Key idea: Put edge between $A \equiv_1 B$ when you can find e s.t. $\varphi_e : A \equiv_1 B$ from a computably bounded search ($\exists e < f(n)$)

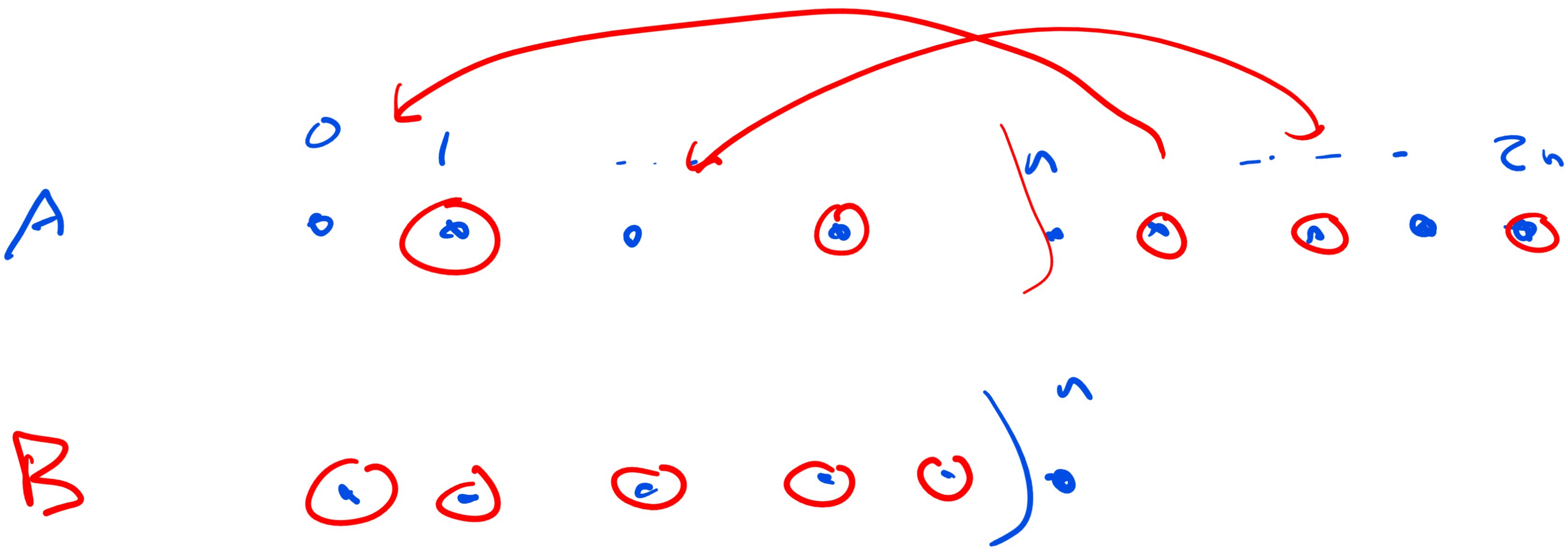


- n is least # > 0 s.t. $B(n-1) \neq B(n)$
- $f(n)$ = max of many indices of computable generated by composing φ_e, $e < n$ (once) with bijection only move #'s $< 2n$

Given $\boxed{\varphi_e}: A \equiv_1 C$ want B with

$$A \overset{G}{\text{———}} B \overset{G}{\text{———}} C$$

Assume A is nontrivial, pick $\underline{n > e}$



Computable Isomorphisms

Today, just focus on <u>relational</u> languages

$\mathcal{L} = \{R_0, R_1, \ldots\}$ is <u>computable</u> if

$n \mapsto \text{arity}(R_n)$ is computable.

$X_{\mathcal{L}}$ = space of $\mathcal{L}$-structures on $\mathbb{N}$

$x \cong^{c}_{\mathcal{L}} y \;:\Longleftrightarrow\;$ x, y are computably isomorphic

Theorem. $\cong^c_{\mathcal{L}}$ is Π^0_2-graphable with diameter 2

Proof follows similar strategy as for $\equiv_1$

For nontrivial $A \subseteq \mathbb{N}$, we were able to "store" information in least n s.t. $A(n-1) = A(n)$

Consider the case of a single binary relation R:

Could be successful using $\equiv_1$-idea with

$$\widetilde{R}(a) :\Longleftrightarrow R(a,a)$$

But, $\tilde{R}(a) \Leftrightarrow R(a,a)$ can be trivial, without R being trivial.

Look for distinct a, b, c s.t.

$$\big(R(a,b) \text{ and } \neg R(a,c)\big)$$

$$\text{or } \big(R(b,a) \text{ and } \neg R(b,a)\big)$$

What if these a, b, c don't exist? R is "trivial"

In general, for n-ary relation R, we should look at all $\tilde{R}$ that can be defined from R and <u>projection</u> <u>functions</u>.

$\Rightarrow$ Call such an $\tilde{R}$ a **shuffle** of R.

Example. $R(a,b,c)$ 3-ary; shuffles of R include

$$R_0(a) :\Longleftrightarrow R(a,a,a)$$

$$R_1(a,b) :\Longleftrightarrow R(b,a,b)$$

$$R_2(a,b,c) :\Longleftrightarrow R(c,b,a)$$

For $x \in X_{\mathcal{L}}$, looking for R^x $(R \in \mathcal{L})$, or one of its shuffles, to have the following coding property:

(∗) there is injective sequence $\vec{a}, b, c$ st.

$$R^x(\vec{a}, b) \text{ and } \neg R^x(\vec{a}, c)$$

What if there are none?

Lemma. If R^x and all of its shuffles fail to have (∗), then R^x is definable by a formula with no non-logical symbols

Friedman - Stanley Jumps

Definition E an eq. rel. on X, then its Friedman - Stanley jump is the eq. rel. E^+ on $X^{\mathbb{N}}$ defined by

$$(x_i) \, E^+ \, (y_i) \; :\Longleftrightarrow \; \{[x_i]_E : i \in \mathbb{N}\} = \{[y_i]_E : i \in \mathbb{N}\}$$

Note. If E is in Γ, then E^+ is in $\forall^{\mathbb{N}} \exists^{\mathbb{N}} \Gamma$.

Theorem (A.) Let X be a "nice" Polish space. Let E be an eq. rel. on X, let Γ be a pointclass that contains Σ^0_1 and is closed under computable substitutions.

If E is Γ-graphable with diameter ℓ, then E^+ is $\forall^{\mathbb{N}}\Gamma$-graphable with diameter $\max(2, \ell)$

Main idea of the proof

Some Questions.

(1) **Diameter question:** For which Γ are there Γ-graphable E which is not Γ-graphable with diameter 2?

(2) **Lower bound questions:** for example,
Is $\equiv_1$ Σ^0_2-graphable?

Thank you!