Continuity of measurable cocycles

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Background on group valued cocycles

The setting for our results is Polish group actions

$$G \curvearrowright X$$

that is, continuous actions by a Polish group G on a Polish space X, i.e., such that the action map $G \times X \to X$ is jointly continuous.

A group valued cocycle is a map

$$G \times X \xrightarrow{\psi} H$$

with values in another Polish group H such that the cocycle equation

$$\psi(gf, x) = \psi(g, fx) \cdot \psi(f, x)$$

holds for all $g, f \in G$ and $x \in X$.

For example, if $X \xrightarrow{\phi} H$ is any function into a Polish group, then the differential

$$d\phi(g,x) = \phi(gx)^{-1}\phi(x)$$

defines a so called trivial cocycle.

Indeed,

$$\begin{aligned} d\phi(gf,x) &= \phi(gfx)^{-1}\phi(x) \\ &= \phi(gfx)^{-1}\phi(fx)\cdot\phi(fx)^{-1}\phi(x) \\ &= d\phi(g,fx)\cdot d\phi(f,x). \end{aligned}$$

But general cocycles need not be trivial.

The principal example of a nontrivial cocycle comes from calculus.

Namely, if we let $\mathrm{Diff}^1(\mathbb{R})$ denote the group of C^1-diffeomorphisms of $\mathbb{R}$, then the chain rule

$$(g \circ f)'(x) = g'\big(f(x)\big) \cdot f'(x)$$

simply expresses that the map

$$(g, x) \in \mathrm{Diff}^1(\mathbb{R}) \times \mathbb{R} \quad \mapsto \quad g'(x) \in \mathbb{R}^\times$$

is a cocycle for the tautological action of $\mathrm{Diff}^1(\mathbb{R})$ on $\mathbb{R}$.

Separate and joint continuity of cocycles

There are several results in the literature regarding the continuity of homomorphisms and cocycles under various notions of measurability.

For example, a long line of results due to Fréchet, Banach, Sierpiński, Steinhaus and Weil state that every Haar measurable homomorphism

$$G \xrightarrow{\pi} H$$

between Polish groups is continuous.

The same statement for Baire category is due to Pettis.

Our main result, however, is most closely related to a recent result by T. Meyerovitch and O. N. Solan (2024).

To state this precisely, suppose $G \times X \xrightarrow{\psi} H$ is a cocycle for a Polish group action $G \curvearrowright X$ with values in a Polish group H.

Then, for fixed $g \in G$ and $x \in X$, we may define the maps

$$X \xrightarrow{\psi_g} H \qquad \text{and} \qquad G \xrightarrow{\psi^x} H$$

by

$$\psi_g(z) = \psi(g, z) \qquad \text{and} \qquad \psi^x(f) = \psi(f, x).$$

We say that ψ is continuous in the second variable if the maps $X \xrightarrow{\psi_g} H$ are continuous for all $g \in G$.

Similarly for the other variable or for notions of measurability.

Theorem (Meyerovitch & Solan)

Suppose G and H are locally compact Polish groups and $G \curvearrowright X$ is a continuous action on a locally compact Polish space X. Assume also that $G \times X \xrightarrow{\psi} H$ is a cocycle with values in a Polish group H, which is continuous in the second variable. If, furthermore,

- *ψ^x is Haar measurable for every $x \in X$*

then ψ is continuous.

Some issues raised by this result:

- Whereas local compactness of G is necessary for the formulation of the result, this is not so for local compactness of neither H nor X. So is this needed? The proof certainly relies on it in an essential way.

- How is the assumption of ψ^x being Haar measurable for every $x \in X$ ensured? One would like to replace this by some type of joint measurability of the map ψ.

Let us remark that the relationship between joint and separate measurability is in general more subtle than that of continuity. Indeed, if a general map

$$G \times X \xrightarrow{\psi} H$$

is continuous, then it is automatically continuous in each variable.

However, if $G \times X \xrightarrow{\psi} H$ is only assumed to be ... measurable, then it need not be ... measurable in the first variable.

- Finally, is there a category version of this result?

We extend the Meyerovitch–Solan result in multiple directions.

Theorem

Let $G \curvearrowright X$ be a Polish group action and suppose that $G \times X \xrightarrow{\psi} H$ is a cocycle with values in a Polish group H. Assume also that ψ is continuous in the second variable. Then ψ is continuous in each of the following cases.

1. *ψ is Baire measurable,*
2. *ψ^x is Baire measurable for a dense set of $x \in X$,*
3. *G is locally compact and ψ^x is Haar measurable for a dense set of $x \in X$,*
4. *G is locally compact with Haar measure λ and there is a fully supported σ-finite Borel measure μ on X such that ψ is $\lambda \times \mu$-measurable.*

Observe, in particular, that case (3),

(3) G is locally compact and ψ^x is Haar measurable for a dense set of $x \in X$,

is a direct extension of Meyerovitch and Solan's theorem discarding the hypothesis of local compactness of H and X and weakening the measurability assumption.

Note also that the measure μ in (4),

(4) G is locally compact with Haar measure λ and there is a fully supported σ-finite Borel measure μ on X such that ψ is $\lambda \times \mu$-measurable,

is not assumed to be invariant or even quasi-invariant under the G-action.

The standing assumption of continuity in the second variable can be slightly weakened.

Indeed, it is enough to assume that

- ψ_g is continuous for all g belonging to a generating set $\Sigma \subseteq G$.

For, if ψ_g and ψ_f are continuous, then so are

$$\psi_{gf} = \psi_g(f\,\cdot)\psi_f(\cdot)$$

and

$$\psi_{g^{-1}} = \big(\psi_g(g^{-1}\,\cdot)\big)^{-1}.$$

For example, it is enough to assume that ψ_g is continuous for a comeagre set of $g \in G$ or, in the case of locally compact groups, a conull set of $g \in G$.

Differentials and polynomials

Fix a Polish group action $G \curvearrowright X$ and a function $X \xrightarrow{\phi} H$.

Recall that the differential cocycle $G \times X \xrightarrow{d\phi} H$ is given by

$$d\phi(g, x) = \phi(gx)^{-1}\phi(x).$$

Thus, for $g \in G$ fixed, the directional differential $X \xrightarrow{d_g\phi} H$ is given as

$$d_g\phi(x) = (d\phi)_g(x) = \phi(gx)^{-1}\phi(x).$$

And similarly, iterated directional differentials can be computed,

$$\begin{aligned} d_g d_f \phi(x) &= d_f\phi(gx)^{-1} d_f\phi(x) \\ &= \phi(gx)^{-1}\phi(fgx)\phi(fx)^{-1}\phi(x). \end{aligned}$$

Thus, as in calculus, we see that the specific dependence on x tends to decrease with the number of differentials taken.

Corollary

Suppose $G \curvearrowright X$ is a transitive Polish group action and

$$X \xrightarrow{\phi} H$$

is a Baire measurable map into a Polish group H. Then the following conditions are equivalent.

1. ϕ *is continuous,*
2. $d\phi$ *is continuous,*
3. $d_g\phi$ *is continuous for all g in a generating set $\Sigma \subseteq G$,*
4. *for every sequence $g_1, g_2, \ldots \in G$, there is a $k \geqslant 0$ such that the iterated differential*

$$d_{g_k} \cdots d_{g_1}\phi$$

is continuous.

Ideas of the proof.

One first show that the differential $d\phi$ is Baire measurable and hence, by our theorem,

$$\begin{aligned} &d\phi \text{ is continuous} \\ &\qquad \Leftrightarrow d_g\phi \text{ is continuous for all } g \text{ in a generating set } \Sigma \subseteq G. \end{aligned}$$

On the other hand, suppose $d\phi$ is continuous with the aim of showing that also ϕ is continuous.

If $x_n \to x$, then, by Effros' theorem, we can find $g_n \in G$ such that

$$g_n \to 1 \qquad \text{and} \qquad x_n = g_n x.$$

By the continuity of $d\phi$, it follows that

$$\begin{aligned} \lim_n \phi(x_n)^{-1}\phi(x) &= \lim_n \phi(g_n x)^{-1}\phi(x) \\ &= \lim_n d\phi(g_n, x) = d\phi(1, x) = 1. \end{aligned}$$

So $\lim_n \phi(x_n) = \phi(x)$ and ϕ is continuous at x. $\square$

Suppose $X \xrightarrow{\phi} H$ is a fixed Baire measurable map.

We observe that there is some flexibility in the equivalence

1. ϕ is continuous,
3. $d_g\phi$ is continuous for all g in a generating set $\Sigma \subseteq G$,

Indeed, all that is assumed is that $G \curvearrowright X$ is a transitive Polish group action. So one may choose an appropriate group for the purpose.

A specific application of this corollary is to so called polynomial maps as defined by A. Leibman.

For $k \geqslant 0$, a map $G \xrightarrow{\phi} H$ between two groups is said to be a polynomial of degree $\leqslant k$ in case

$$d_{g_k} \cdots d_{g_0}\phi \;\equiv\; 1_H$$

for all $g_0, \ldots, g_k \in G$. Here the action $G \curvearrowright G$ is left-multiplication.

More generally, we may define $G \xrightarrow{\phi} H$ to be a polynomial of potentially transfinite degree provided that, for all $g_0, g_1, g_2, \ldots \in G$, there is a k such that

$$d_{g_k} \cdots d_{g_0}\phi \;\equiv\; 1_H.$$

- ϕ has degree $\leqslant 0$ if and only if ϕ is a constant map,
- ϕ has degree $\leqslant 1$ if and only if $\phi(1)\phi(\,\cdot\,)^{-1}$ is a group homomorphism.

Meyerovitch and Solan showed that Haar measurable polynomial maps $G \to H$ of finite degree between locally compact Polish groups are continuous.

We have the same result for category and can also discard local compactness assumption on H in their result.

Corollary

Every Baire measurable polynomial map between Polish groups is continuous.