Borel equivalence relations and forcing

Dima Sinapova
Rutgers University
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joint work with F. Calderoni

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(The Union Problem): Does hyperhyperfinite imply hyperfinite?

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More general examples:

- Def. Let G be a countable group acting on a space X in a Borel way. The induced orbit equivalence relation E_G is a cber.
- And actually every cber is obtained in such a way (Feldman Moore)

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Theorem (Smythe)

1. $\text{Gen}(\mathbb{P}, M)$ is a G_δ subset of $2^{\mathbb{P}}$.
2. $E^M_{\mathbb{P}}$ is a countable Borel equivalence relation.
3. $E^M_{\mathbb{P}}$ is induced by the action of the group of automorphisms of the boolean completion of $\mathbb{P}$ that are in M.

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We show a characterization of smoothness for equivalence relations of the form $E^M_{\mathbb{P}}$.

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The idea: a translation of the topological characterization of smoothness via condensation, which is weaker than generic ergodicity.

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So, an M-generic filter G is a condensed point if for all p with $p \in G$, there is another M -generic filter H with $p \in H$, such that $M[G] = M[H]$ and $G \neq H$.

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idea of the proof: take lottery sums of nonisomorphic homogeneous forcings in a tree like fashion.

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For $p = \langle s, A\rangle \in \mathbb{P}$, set $lh(p) = |s|$.

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Theorem. $E^M_{\mathbb{P}}$ is hyperfinite.

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Then, for comeagerly many x, there are comegearly many such y's.

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Lemma

If y and x are M-mutually generic, and $\{g_n \mid n < \omega\} \in M[y]$*, then* $[x]_E = [x]_{E_y}$*. Namely,* $E^M_{\mathbb{P}}$ *restricted to* $Gen(\mathbb{P}, M[y])$ *is hyperfinite.*

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Let $\mathrm{N} \supset \mathrm{M}$ be a countable model, such that the $\{\mathrm{g_n} \mid \mathrm{n} < \omega\} \in \mathrm{N}$. Define $\mathrm{E^N}$ by $\mathrm{xE^Nz}$ iff:

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Let $N \supset M$ be a countable model, such that the $\{g_n \mid n < \omega\} \in N$.
Define E^N by xE^Nz iff:
xEz, $x \restriction \mathrm{Even} = z \restriction \mathrm{Even}$, and the latter is N -generic.

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Lemma

If y and x are M-mutually generic, and $\{g_n \mid n < \omega\} \in M[y]$, *then* $[x]_E = [x]_{E_y}$. *Namely,* $E_{\mathbb{P}}^M$ *restricted to* $Gen(\mathbb{P}, M[y])$ *is hyperfinite.*

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E^N *is hyperfinite.*

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- show it is hyperfinite when $V = L$.

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Idea:

- show it is hyperfinite when $V = L$.
- then use absoluteness.

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6. What other forcings besides the Random real are not hyperfinite?

THANK YOU