Measurable 2-factors in regular bipartite graphs

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Part I

Overview

Matchings (and related notions) play a central role in graph theory and its applications.

We review some basic concepts and history, and survey some more recent developments.

We will eventually generalize a bit to discuss k-factors in addition to matchings.

This includes joint work with Matt Bowen, Alekos Kechris, Ben Miller, and Felix Weilacher.

Part II

Definitions and background

Definitions

A *graph* G on a (probably infinite) set X is a symmetric, irreflexive subset of X^2. We colloquially refer to elements of X as *vertices* and (symmetrized) elements of G as *edges*.

Such a graph G is *bipartite* if there is a partition $X = A \sqcup B$ so that every edge is between some element of A and some element of B.

A *matching* in a graph G is a subgraph $M \subseteq G$ for which every vertex is incident with at most one edge of M.

The vertices incident with an edge of M will be called its *support*.

The matching M is *perfect* if its support is X.

Theorem (Hall, 1935)

A locally finite bipartite graph admits a perfect matching if and only if it satisfies Hall's condition.

Definition

A locally finite graph G on X satisfies *Hall's condition* if for all finite $C \subseteq X$, $|N_G(C)| \geq |C|$.

Remark

Note that a locally finite bipartite graph in which every vertex is incident to the same number of edges automatically satisfies Hall's condition. Such graphs are called (d-)*regular*.

Part III

Descriptive combinatorics is born

Lazckovich identified various equidecomposition tasks with the search for perfect matchings. Using Hall's Theorem, he resolved Tarski's circle-squaring problem.

Theorem (Laczkovich, 1990)

A disc and square of the same area are equidecomposable via (finitely many) translations of the plane.

Remark

The argument, ultimately relying upon compactness, produces set-theoretically hideous pieces. Laczkovich realized that solving the problem with nicer pieces corresponds to finding matchings with better regularity properties.

Question

Can we get measurable matchings out of Hall's theorem?

Answer

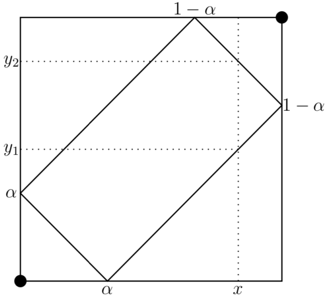
Not in general, even when the graph itself is very nice.

III. Descriptive combinatorics is born

Graphs with no measurable matching

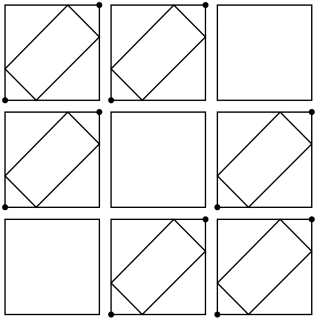
Example (Laczkovich, 1988)

This 2-regular bipartite graph has no measurable perfect matching:



Example (with Kechris, 2013)

This can be tweaked into a $2n$-regular graph with no measurable perfect matching:



Remark

Finding such examples with odd degree seems much harder.

Theorem (Kun, 2025+)

For every $d \geq 2$, there is a measure-preserving d-regular acyclic Borel graph on a standard probability space (X, μ) admitting no μ-measurable perfect matching.

Remark

Kun's examples are highly *non*-hyperfinite, while the Laczkovich-style examples are hyperfinite. Is there any hope of finding a 3-regular acyclic (or just bipartite) *hyperfinite* Borel graph with no measurable matching?

Part IV

Hyperfinite graphs

From now on we assume that X is a Polish space equipped with a Borel probability measure μ.

Remark

You don't really lose anything by assuming that X is the unit interval and that μ is Lebesgue measure.

Remark

We also assume that G is Borel (as a subset of X^2), and investigate Borel matchings. If such a matching's support is conull (or comeager), we can say we've found a Borel perfect matching mod null (resp., mod meager).

Definition

We say that a graph is *component-finite* if every connected component is finite.

Definition

We say that a graph is *hyperfinite* if it is an increasing union of component-finite Borel graphs.

Remark

Hyperfinite graphs are “well approximated” by finite graphs, so one might hope they behave much like finite graphs. This hope is false in the Borel context, but is generally true mod null or meager.

Remark

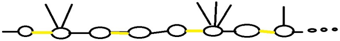
Every locally countable Borel graph is hyperfinite mod meager.

Definition

A *G-ray* is an injective sequence (x_n) of vertices with every $(x_n, x_{n+1}) \in G$.

Definition

A G-ray is *bad* if every even-indexed vertex of the sequence has degree 2.



Theorem (with Miller, 2017)

Suppose that G is an acyclic hyperfinite Borel graph with no bad G-rays. Then G admits a Borel matching whose support contains every vertex of degree at least 2, mod null or meager.

Corollary

Suppose that G is an acyclic hyperfinite Borel graph and that every vertex has degree at least 3. Then G admits a Borel perfect matching, mod null or meager.

Remark

In particular, there can be no 3-regular acyclic example like Kun's that is also hyperfinite.

Question

What about more general bipartite graphs?

Theorem (Bowen-Kun-Sabok, 2021)

Suppose that G is a one-ended regular bipartite hyperfinite Borel graph on (X, μ). Moreover, assume that μ is G-invariant. Then G admits a perfect matching mod null.

Theorem (Bowen-Poulin-Zomback, 2022)

Suppose that G is a one-ended regular bipartite hyperfinite Borel graph on X. Then G admits a perfect matching mod meager.

Remark

In fact, the above work establishes the more general existence of *k-factors*, which are k-regular subgraphs of G. So a 1-factor is just a perfect matching.

Remark

Both arguments proceed by "rounding cycles" in a fractional k-factor to obtain an integer-valued one.

Theorem (with Bowen and Weilacher, 2025+)

Suppose that $d \geq 2$ and that G is a d-regular bipartite hyperfinite Borel graph on (X, μ). Then G admits a Borel 2-factor mod null or meager.

Corollary

In particular, when d is odd one can iteratively pull off 2-factors and obtain a perfect matching! This finally rules out all bipartite hyperfinite versions of Kun's example.

Remark

In a soft sense, this means among regular bipartite hyperfinite graphs Laczkovich's example is the only one.

Part V

Sketch of the proof

Definition

Let $I = [0, 1]$ denote the usual unit interval. For positive $n \in \mathbb{N}$, let $I_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ denote the $\frac{1}{n}$-*discretized* interval.

Definition

Given a graph G on X and $k \in \mathbb{N}$, a *fractional k-factor* of G is a symmetric function $f : G \to I$ satisfying for all $x \in X$,

$$\sum_{xGy} f(x, y) = k.$$

Remark

It makes sense to ask about Borel fractional k-factors, etc., in the usual fashion.

Lemma

Suppose that G is a bipartite hyperfinite Borel graph on (X, μ) and that k, d are positive naturals. Suppose further that G admits a Borel I_d-valued fractional matching. Then G admits a Borel $(I_2 \cap I_d)$-valued fractional matching mod null or meager.

Remark

$I_2 \cap I_d = \{0, 1\}$ when d is odd, while
$I_2 \cap I_d = \{0, \frac{1}{2}, 1\}$ when d is even.

Remark

So this lemma grants a Borel perfect matching mod null or meager whenever d is odd and G is a d-regular hyperfinite Borel graph..

Proof

- First, since all cycles have even length, we can round them away (mod null or meager) to build an I_d-valued fractional matching f so that the subgraph

$$H = \{(x, y) : f(x, y) > 0\}$$

  is acyclic.
- Next, check that the fractional matching forces every bad H-ray to be eventually 2-regular.
- Then the previous result with Miller grants a Borel matching mod null or meager, except possibly on some 2-regular components of H on which f is constantly $1/2$. This is the type of fractional matching we want. $\square$

Theorem (with Bowen and Weilacher, 2025+)

Suppose that $d \geq 2$ and that G is a d-regular bipartite hyperfinite Borel graph on (X, μ). Then G admits a Borel 2-factor mod null or meager.

Remark

The odd d case reduces to the even d case by stripping away a perfect matching, so we will focus on the case $d = 4$ for convenience.

Remark

The main idea is to strategically reweight a fractional matching and "induce mitosis."

Sketch of the proof

The lemma grants a Borel I_2-valued Borel fractional matching f mod null or meager. We build an I_3-valued fractional 2-factor g by declaring

$$g = \frac{1}{3} \text{ when } f = 0$$
$$g = \frac{2}{3} \text{ when } f = \frac{1}{2}$$
$$g = 1 \text{ when } f = 1.$$

This generalizes to other even d with enough arithmetic.

Sketch of the proof, cont.

- Now define a graph H on $2 \times X$ that injectively projects to G by letting each of $(0, x)$ and $(1, x)$ claim a set of G-edges whose g-values sum to 1.
- As H admits an I_3-valued fractional matching, another application of the rounding lemma grants a perfect matching of H mod null or meager.
- Projecting this matching back to G yields the desired 2-factor mod null or meager.

$\square$

Part VI

Thanks!