Continuous Hyperfiniteness

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Preliminary Descriptive Set Theory

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Examples

$\mathbb{R}, \mathbb{C}, C[0,1], L^p, l^p (1 \leq p < \infty), 2^\omega$(Cantor space)$, \omega^\omega$(Baire space)

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$xE_0y \Leftrightarrow (\exists m)(\forall n > m)\ x(n) = y(n)$.
Then E_0 is hyperfinite.

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- E_G^X is the induced orbit equivalence relation of the action of G on X. i.e., $xE_G^X y \Leftrightarrow \exists g \in G \; g \cdot x = y$.
- From this, Kechris et al. expanded the theory of countable Borel equivalence relations, which has since become a thriving area.

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Comparing the Borel complexities of various Borel equivalence relations has recently been a booming area.

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- Schneider and Steward (2013) proved when G is locally nilpotent.
- Conley, Jackson, Marks, Seward and Tucker-Drob (2020) proved when G is polycyclic.

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Theorem

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1. E is (Borel) hyperfinite.
2. E is Borel liminf finite (liminf of a sequence of finite Borel equivalence relations).
3. E is induced by a Borel action of $\mathbb{Z}$ on X.
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Proof.

(1 $\Leftrightarrow$ 2) immediate (1 $\Leftrightarrow$ 3) Weiss, Slaman and Steel (1 $\Leftrightarrow$ 4 $\Leftrightarrow$ 5) by Dougherty, Jackson and Kechris □

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A topological space X is **zero dimensional** if it admits a clopen basis. Let a group G act on X. Then we say the action is **continuous** if the action as a map from $G \times X$ to X is continuous.

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2^ω, ω^ω are zero-dimensional second countable Hausdorff spaces (in fact, they are Polish).

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Examples

2^ω, ω^ω are zero-dimensional second countable Hausdorff spaces (in fact, they are Polish).
If G is a countable discrete group, then the shift action of G on 2^G (which is zero-dimensional second countable Hausdorff space) is continuous, where the action is $(g \cdot x)(h) = x(g^{-1}h)$

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Let X be a zero dimensional second countable Hausdorff space. Let G be a countable discrete group continuously acting on X. Then is it true that

1. E_G^X is continuously G-hyperfinite (increasing union of a sequence of finite G-clopen equivalence relations on its field)
2. E_G^X is continuously G-liminf finite (liminf of a sequence of finite G-clopen equivalence relations on its field)
3. E_G^X is induced by a continuous action of $\mathbb{Z}$ on X.
4. $E_G^X \sqsubseteq_c E_0$
5. $E_G^X \leq_c E_0$

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- (1) is true when G acts on itself $(X = G)$.
- We show (1) is false when G is finitely generated, X is compact and admits an hyperaperiodic element (which is true when $X = 2^G$ and the action is shift by Gao, Jackson, Seward (2016)).

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Let X be a zero dimensional second countable Hausdorff space with a countable discrete group G continuously acting on X.

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- We show (1) is false when G is finitely generated, X is compact and admits an hyperaperiodic element (which is true when $X = 2^G$ and the action is shift by Gao, Jackson, Seward (2016)). By similar proof, we can show under the same condition, there is no (G-)continuous toast on X.

Borel vs Continuous Weiss Question

Definition

Let G be a finitely generated group with a finite generator S (without identity) acting on a topological space X.

- $\{T_n\}$ is an (unlayered) toast on X iff
  1. For any n, T_n is a finite equivalence relation on its field and $T_n \subseteq E_G^X$.
  2. $\bigcup_{n\in\omega} \text{Field}(T_n) = X$
  3. (hit or miss property) For any $n < m$, T_n-class C and T_m-class C', either C and C' are disjoint or $C \subseteq C'$.
  4. For any $n \in \omega$ and T_n-class C, there is $m > n$ and a T_m-class C' such that $C \subseteq C' \setminus \partial C'$.

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- $\{T_n\}$ is a layered toast on X iff $\{T_n\}$ is an unlayered toast and For any $n \in \omega$ and T_n-class C, there is a T_{n+1}-class C' such that $C \subseteq C' \setminus \partial C'$.

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- For a toast $\{T_n\}$ on X,
  - $\{T_n\}$ is Borel iff for any $n \in \omega$, T_n is Borel subset of $X \times X$.
  - $\{T_n\}$ is (G-)continuous iff for any $n \in \omega$, T_n is G-clopen in X.

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Figure: Layered toast

Borel vs Continuous Weiss Question

Question

Let G be a finitely generated group shift acting on 2^G. Then

1. (Continuous toast cover) Is there a countable G_δ cover $\mathcal{B}$ of 2^G such that any $B \in \mathcal{B}$ is a continuous toast?
2. (Borel toast cover) Is there a countable Borel cover $\mathcal{B}$ of 2^G such that any $B \in \mathcal{B}$ is a Borel toast?

Borel vs Continuous Weiss Question

Definition

Let $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_2$ be definabilities and let κ, μ be cardinals. Then $G \curvearrowright X$ has the μ-size $\mathbf{\Gamma}_1$-piecewise $\mathbf{\Gamma}_2$-chromatic number $\leq \kappa$ if there is a $\mathbf{\Gamma}_1$ cover of X with size μ such that each element of cover has the $\mathbf{\Gamma}_2$-chromatic number $\leq \kappa$.

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Folklore

Continuous chromatic number of $\mathbb{Z}^n$ shift acting on $F(2^{\mathbb{Z}^n})$ is 3 when $n = 1$ and 4 when $n \geq 2$.

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Theorem (Jackson, Kang)

$\mathbb{Z}^n$ shift acting on $F(2^{\mathbb{Z}^n})$ has the finite-size G_δ-piecewise continuous chromatic number 3.

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- (Conjecture) (2) is false for any G shift acting on $X = 2^G$.

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Lemma

Let X be a second countable Hausdorff space with a countable discrete group G continuously acting on X. Then for any $d \in \omega$ the following are equivalent:

1. For any finite $A \subseteq G$ there is a bounded G-clopen equivalence relation E on X such that for any $x \in X$, $B(x; A)$ meets at most $d + 1$-many E-equivalence classes.
2. For any finite $A \subseteq G$ there is a clopen covering $\{V_0, V_1, \cdots, V_d\}$ of X such that for any $i = 0, 1, \cdots, d$, $\mathfrak{F}_A(V_i)$ is G-clopen finite.

If any one of the above holds, we say $\mathrm{asi}_c(G \curvearrowright X) \leq d$.

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Proof.

Conley, Jackson, Marks, Seward and Tucker-Drob (2020) modulo Borel/clopen argument □

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- (4) (Continuous embedding problem) is true when $G = \mathbb{Z}^n$ ($n = 1$ by Boykin and Jackson, later extended to arbitrary n by Gao and Jackson).

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- We show (4) is true when G is locally finite.
- (Question) Is (4) true for G abelian? Equivalently, when $G = \mathbb{Z}^{<\omega}$?

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- General Burnside Problem is equivalent to: If G is a torsion group, then it is locally finite.
- By Ching Chou, General Burnside Problem holds for any elementary amenable group.
- Therefore, (4) is true when G is torsion elementary amenable group.

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Definition

Let X be a zero dimensional second countable Hausdorff space with a countable discrete group G continuously acting on X. Then the **continuous asymtotic dimension** of G acting on X is less than equal to $d \in \omega$ if for any finite $A \subseteq G$, there is a clopen cover $\{V_0, V_1, \cdots, V_d\}$ such that for any $i = 0, 1, \cdots d$, $\mathfrak{F}_A(V_i)$ is G-clopen uniformly bounded

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Definition

Let X be a zero dimensional second countable Hausdorff space with a countable discrete group G continuously acting on X. Then the **continuous asymtotic dimension** of G acting on X is less than equal to $d \in \omega$ if for any finite $A \subseteq G$, there is a clopen cover $\{V_0, V_1, \cdots, V_d\}$ such that for any $i = 0, 1, \cdots d$, $\mathfrak{F}_A(V_i)$ is G-clopen uniformly bounded, where an equivalence relation E is G-clopen if for any $g \in G$ the set $\{x : xEg \cdot x\}$ is clopen.

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The definition of G-clopenness was first introduced by Gao and Jackson in "Countable abelian group actions and hyperfinite equivalence relations" with $G = \mathbb{Z}^n$.

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Proof.

Go back to the proofs of "Borel asymptotic dimension and hyperfinite equivalence relations" by Conley, Jackson, Marks, Seward and Tucker-Drob with considering G-clopen instead of Borel! □

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$$\delta(g) = \begin{cases} 1 \text{ (If } g = a^{-n} \text{ for some } n) \\ 0 \text{ (If otherwise)} \end{cases}.$$

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. Fix large m so that A is disjoint from $\{a^{-m-1}, a^{-m-2}, \cdots\}$. Then $a^m \cdot \delta = 0$ on A so that $a^m \cdot \delta E a^{m+1} \cdot \delta$, which is a contradiction. □

(2) implies (4)

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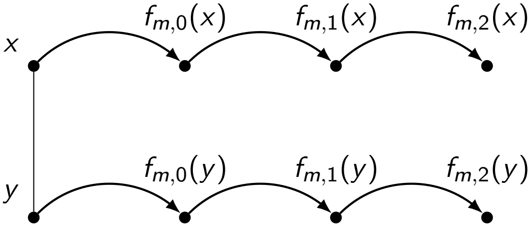

Figure: Continuously moving the points

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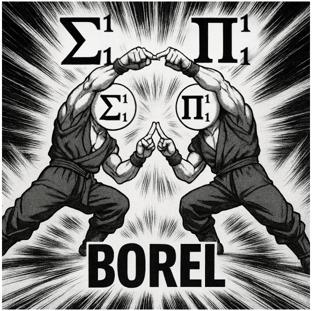

Figure: Suslin's Theorem

Thank you!