Tarski's circle squaring problem with algebraic translations and few pieces

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Thm: A closed disc and closed square of the same area in $\mathbb{R}^2$ can be equidecomposed using 9216 pieces and algebraic translations.

Equidecomposition in $\mathbb{R}^n$

If $a\colon \Gamma \curvearrowright X$ is a group action, $A, B \subseteq X$ are a-**equidecomposable** if there exist a finite partition $A = A_1 \sqcup \ldots \sqcup A_n$ of A and $\gamma_1, \ldots, \gamma_n \in \Gamma$ so $B = \gamma_1 \cdot_a A_1 \sqcup \ldots \gamma_n \cdot_a A_n$.

- Banach-Tarski 1924 (AC): A unit ball and a union of two disjoint unit balls in $\mathbb{R}^3$ are equidecomposable by isometries.
- Tarski 1929 (AC): If $a\colon \Gamma \curvearrowright X$ there is a finitely additive a-invariant measure on all subsets of X (i.e. a is amenable) iff there does not exist a partition $X = A \sqcup B$ so that X is a-equidecomposable with A and X is a-equidecomposable with B (i.e. a is not paradoxical). *Proof.* Hall-Rado matching. $\square$
- No Banach-Tarski in $\mathbb{R}^2$: There is no equidecomposition of a unit ball and two unit balls in $\mathbb{R}^2$. *Proof.* There is a finitely additive isometry-invariant extension of Lebesgue measure to all subsets of $\mathbb{R}^2$, since $\mathrm{Isom}(\mathbb{R}^2)$ is amenable. $\square$
- **Thm Laczkovich (1990), answering Tarski (1925)**: A closed disc and closed square of the same area in $\mathbb{R}^2$ are equidecomposable by translations.

Laczkovich's general equidecomposition theorem

Thm (Laczkovich 1992): Suppose $A, B \subseteq \mathbb{R}^k$ are bounded sets with $\overline{\dim}_{\mathrm{box}}(\partial A), \overline{\dim}_{\mathrm{box}}(\partial B) < k$ and $\lambda(A) = \lambda(B) > 0$, then A and B are equidecomposable by translations.

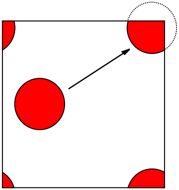
$\partial A := \mathrm{cl}(A) \setminus \mathrm{int}(A)$ is the topological boundary of A, and λ is Lebesgue measure. E.g. if A is a closed disc, then $\overline{\dim}_{\mathrm{box}}(\partial A) = 1$

Laczkovich's theorem uses a beautiful collection of ingredients: compactness arguments to turn infinite into finite combinatorics, quantitative bounds on equidistribution, Fourier analysis, and Diophantine approximation.

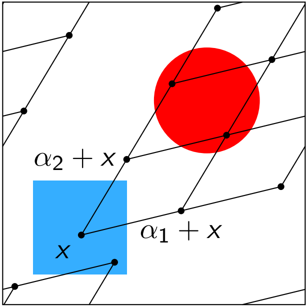
Our paper gives a new proof of this theorem that tries to be as simple and self-contained as possible: "A new proof of Laczkovich's circle squaring theorem I". We'd greatly appreciate any feedback – we want it to be as readable and accessible as possible (and accessible to undergraduates)!

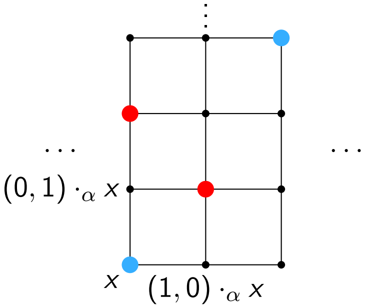
Proof of Laczkovich's theorem. Step 1: work in the torus

Scale the closed disc and square $A, B \subseteq \mathbb{R}^2$ to lie in $[0,1)^2$. Then A, B are equidecomposable by translations in $\mathbb{R}^2$ iff they are equidecomposable by translations in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.



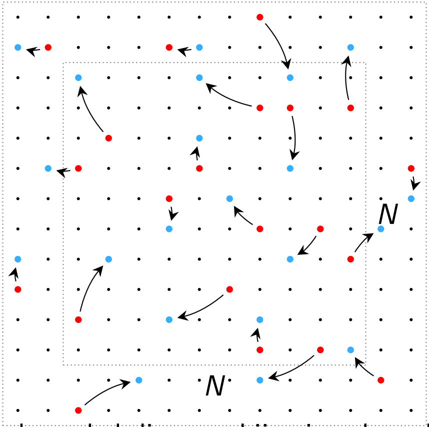
d translations $\alpha = (\alpha_1, \ldots, \alpha_d)$ give an action of $\mathbb{Z}^d$ on $\mathbb{T}^2$.





Equidecomposition gives bounds on equidistribution

There is an equidecomposition of A and B in this action iff there is a bounded distance bijection in the associated Schreier graph.



Now if there is a bounded distance bijection then there exists a constant c so that for all $N \in \mathbb{N}$ and all $x \in \mathbb{T}^2$,

$$\left| |\{0, \ldots, N-1\}^d \cdot_\alpha x \cap A| - |\{0, \ldots, N-1\}^d \cdot_\alpha x \cap B| \right| \leq cN^{d-1}.$$

Equidistribution gives equidecomposition

Let $F_N(x, \alpha) = \{0, \ldots, N-1\}^d \cdot_\alpha x = \{n_1\alpha_1 + \ldots + n_d + x\alpha_d \colon 0 \leq n_i < N\}$. By the ergodic theorem, expect $|F_N(x, \alpha) \cap A| \approx |F_N(x, \alpha)|\lambda(A)$.

If F is a finite set, let $D(F, A) = \left| \frac{|F \cap A|}{|F|} - \lambda(A) \right|$ be the **discrepancy** of F with respect to A.

Laczkovich proves "almost" a converse to the previous observation.

Lemma A (Laczkovich): If $\exists c, \delta > 0$ such that $D(F_N(x, \alpha), A), D(F_N(x, \alpha), B) \leq cN^{-1-\delta}$, then A and B are equidecomposable in the translation action given by α.
Proof. Clever counting argument using Hall-Rado. $\square$

Lemma B (Laczkovich): There exists $\alpha = (\alpha_1, \ldots, \alpha_d)$ so $D(F_N(x, \alpha), A), D(F_N(x, \alpha), B) \leq cN^{-1-\delta}$.
Proof. Almost every α works if d large enough. Use the Erdős-Turán-Koksma inequality and metric diophantine approx. $\square$

Our new idea: use product actions

Recall if $F \subseteq \mathbb{T}^k$ is finite and $A \subseteq \mathbb{T}^k$ then $D(F, A) = \left| \frac{|F \cap A|}{|F|} - \lambda(A) \right|$.

Instead of using a random actions in Lemma B, we can instead use actions of $\mathbb{Z}^{kd} \curvearrowright \mathbb{T}^k$ that are products of one-dimensional actions $\mathbb{Z}^d \curvearrowright \mathbb{T}$. If $\alpha \colon \mathbb{Z}^{kd} \curvearrowright \mathbb{T}^k$ is a product action, then $F_N(x, \alpha)$ is a product of 1-dimensional sets.

If $F \subseteq \mathbb{T}$ is finite, let $D(F) = \sup_{\text{intervals } I} D(F, I)$. In one dimension $D(F)$ can be bounded using the simpler Erdős-Turán inequality (simpler than Erdős-Turán-Koksma) where it is easier to bound the results.

Then one can "lift" upper bounds on $D(F_i)$ to bounds on $D(\prod_i F_i, A)$ when $A \subseteq \mathbb{T}^k$ has $\overline{\dim}_{\text{box}}(\partial A) < k$.

Example lifting argument for convex sets

$D(F, A) = \left| \frac{|F \cap A|}{|F|} - \lambda(A) \right|$ and $D(F) = \sup_{\text{intervals } I} D(F, I)$.

Lemma: If $A \subseteq [0,1)^2$ is convex, and $F_1, F_2 \subseteq \mathbb{T}$ are finite, then $D(F_1 \times F_2, A) \leq D(F_1) + D(F_2)$.

Proof. A_x and A^y are convex for all x, y. Let λ^1 and λ^2 be Lebesgue measure on the two copies of $\mathbb{T}$. Let μ_i be the uniform probability measure supported on F_i. Note that $D(F_i) = \sup_{\text{intervals I}} |\mu_i(I) - \lambda(I)|$.

$$
\begin{aligned}
& D(F_1 \times F_2, A) \\
& = |\mu_1 \times \mu_2(A) - \lambda^1 \times \lambda^2(A)| \\
& \leq |\mu_1 \times \mu_2(A) - \mu_1 \times \lambda^2(A)| + |\mu_1 \times \lambda^2(A) - \lambda^1 \times \lambda^2(A)| \\
& \leq \int |\mu_2(A_x) - \lambda^2(A_x)| \, \mathrm{d}\mu_1(x) + \int |\mu_1(A^y) - \lambda^1(A^y)| \, \mathrm{d}\lambda^2(y) \\
& \leq D(F_2) + D(F_1)
\end{aligned}
$$

since $|\mu_2(A_x) - \lambda^2(A_x)| \leq D(F_2)$ and $|\mu_1(A^y) - \lambda^1(A^y)| \leq D(F_1)$ for all x, y. $\square$

Finishing the proof

Thm (Erdős-Turán inequality): There is an absolute constant C so that if $F \subseteq \mathbb{T}$, then for any number m,

$$D(F) \leq C\left(\frac{1}{m+1} + \sum_{k=1}^{m} \frac{1}{k|F|}\left|\sum_{x \in F} e^{2\pi i k x}\right|\right)$$

For $F_N(x, \alpha) = \{n_1\alpha_1 + \ldots + n_d\alpha_d + x \colon 0 \leq n_i < N\}$, letting $\|x\|$ denote the distance from x to the nearest integer, summing geometric series and using $|\sin(\pi x)| \geq 2\|x\|$ we get

$$D(F_N(x, \alpha)) \leq C\left(\frac{1}{m+1} + \frac{1}{2^d N^d}\sum_{k=1}^{m} \frac{1}{k\|k\alpha_1\| \cdots \|k\alpha_d\|}\right)$$

These types of sums

$$S_N(\alpha_1, \ldots, \alpha_d) = \sum_{n=1}^{N} \frac{1}{n\|n\alpha_1\| \cdots \|n\alpha_d\|}$$

are studied in the number theory literature. Are related e.g. to the Littlewood conjecture: $\liminf_{n \to \infty} n\|n\alpha_1\|\|n\alpha_2\| = 0$ for all α_1, α_2.

Putting everything together

Thm (M.-Unger): Suppose $\epsilon > 0$ and $A, B \subseteq \mathbb{R}^k$ are bounded Lebesgue measurable sets such that $\lambda(A) = \lambda(B) > 0$, and $\overline{\dim}_{\mathrm{box}}(\partial A) \leq k - \epsilon$ and $\overline{\dim}_{\mathrm{box}}(\partial B) \leq k - \epsilon$. Suppose $c > 0$ and $1, \alpha_1, \ldots, \alpha_d \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$ so for all N

$$S_N(\alpha_1, \ldots, \alpha_d) = \sum_{n=1}^{N} \frac{1}{n\|n\alpha_1\| \cdots \|n\alpha_d\|} \ll_u N^c,$$

and $c < d\epsilon - 1$. Then A and B are equidecomposable in $\mathbb{R}^k$ by finitely many translations whose coordinates are integer linear combinations of $1, \alpha_1, \ldots, \alpha_d$.

Diophantine approximation

How well can a real number be approximated by rationals? (Motivation: precise correspondences between how well α is approximated by rationals, and how quickly ergodic averages converge in the translation action by α in $\mathbb{R}/\mathbb{Z}$).

Thm (Dirichlet): For every real number α, there are infinitely many integers p and q so that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$. Equivalently, for infinitely many q, $\|q\alpha\| < q^{-1}$. *Proof.* Pigeonhole principle. $\square$

Thm (Roth, 1955): If α is an algebraic number, then for every $\epsilon > 0$ there are only finitely many q so that: $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$. Equivalently, there is a constant $c_{\alpha,\epsilon}$ so $\|\alpha q\| \geq c_{\alpha,\epsilon} q^{-1-\epsilon}$ for all q.

Thm (Schmidt 1970): If $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$ are algebraic irrationals that are linearly independent over $\mathbb{Q}$, then for every $\epsilon > 0$, there is a constant c so that for every integer $q > 0$, $\|q\alpha_1\| \cdots \|q\alpha_d\| \geq c q^{-1-\epsilon}$.

Equidecompositions with algebraic irrationals

Immediate corollary of Schmidt's theorem: if $1, \alpha_1, \ldots, \alpha_d$ are algebraic numbers linearly independent over $\mathbb{Q}$, then for all $\epsilon > 0$, $S_N(\alpha_1, \ldots, \alpha_d) \ll N^{1+\epsilon}$.

Cor: Laczkovich's equidecompositions can be done with algebraic translations. E.g. a closed disc and square are equidecomposable in an action of $\mathbb{Z}^6 \curvearrowright \mathbb{T}^2$ by algebraic translations.

This answers a 1990 problem of Laczkovich.

Thm (M.-Unger, using an idea and Lemma of Calegari): If $1, \alpha_1, \ldots, \alpha_d$ are algebraic numbers that are linearly independent over $\mathbb{Q}$, then for any $\epsilon > 0$,

$$S_N(\alpha_1, \ldots, \alpha_d) \ll N^{1-\frac{1}{d}+\epsilon}.$$

Cor: a closed disc and square are equidecomposable in an action of $\mathbb{Z}^4 \curvearrowright \mathbb{T}^2$ by algebraic translations.
Open: are a closed disc and square are equidecomposable in an action of $\mathbb{Z}^2 \curvearrowright \mathbb{T}^2$ by algebraic translations?

Bounding the number of pieces to square the circle

The bound on the number of pieces comes from a bound on a certain flow from the circle to the square. This flow in turn comes from an infinite summation over $S_N(\alpha_1, \alpha_2)$ roughly of the form $\sum_{N=1}^{\infty} N^{-2} S_N(\alpha_1, \alpha_2)$. So from effective bounds on such sums we can get effective bounds on the number of pieces.

Recall **Roth's Thm**: for all algebraic α and $\epsilon > 0$ there exists $c_{\alpha,\epsilon} > 0$ so that $\|q\alpha\| \geq c_{\alpha_\epsilon} q^{-1-\epsilon}$ for all integers $q > 0$.

Open: is there an algorithm that can calculate $c_{\alpha,\epsilon}$ for a given algebraic α and $\epsilon > 0$?

Progress for $\alpha = \sqrt[3]{2}$.

- Baker (1964) $\|q\sqrt[3]{2}\| \geq 10^{-6} q^{-1.955}$
  (first progress on effective versions of Roth's theorem)
- Easton (1986) $\|q\sqrt[3]{2}\| \geq 2.2 \cdot 10^{-8} q^{-1.795}$.
- Korobov (1990), Bennett (1996) $\|q\sqrt[3]{2}\| \geq 0.25 q^{-1.5}$.
- Voutier (2007) $\|q\sqrt[3]{2}\| \geq 0.25 q^{-1.4325}$.

Effective bounds and computer assistance

We need to bound $\sum_{N=1}^{\infty} N^{-2} S_N(\alpha_1, \alpha_2)$ for some algebraic irrational α_1, α_2. Take $\alpha_1 = \sqrt[3]{2}$ and $\alpha_2 = \sqrt[3]{4}$. We can bound their individual rational approximations to α_i using Voutier, and effectively control their simultaneous approximations since they are linearly independent over $\mathbb{Q}$ and lie in a number field of degree 3.

Thm (M-U): $\sum_{n=1}^{N} \frac{1}{n\|n\sqrt[3]{2}\|\|n\sqrt[3]{4}\|} \leq 1575.42 N^{0.9325}(\log N + 1)$

Gives circle-squaring with 10^{31} pieces. The best previous bound: 10^{40} (Laczkovich).

Obvious improvement: compute these sums $S_N(\alpha_1, \alpha_2)$ exactly for small N, and use the upper bound only once N is large enough that the remaining infinite tail is small. This doesn't work because the series converges too slowly.

Actual method: break the infinite sum $\sum_{N=1}^{\infty} N^{-2} S_N(\alpha_1, \alpha_2)$ into 7 different intervals, using a different algorithm for each to try and get a bound as accurate as possible.

Thm (M.-Unger): The circle can be squared with 9,216 pieces.

Ingredients in our computer assisted proof

- A new "axis-parallel" flow
- Voutier's effective exponents of irrationality.
- An effective version of Khintchine's transference principle, and Laczkovich's method for bounding $S_N(\alpha_1, \alpha_2)$ from bounds on simultaneous approximations to α_1, α_2.
- The Bombieri-van der Poorten-Shiu fast algorithm for continued fraction expansions of algebraic numbers.
- Slater's 3-gap theorem: in an irrational rotation in $\mathbb{T}$ by α, the return times to an interval $[0, b)$ have three possible values which can be effectively calculated from the continued fraction expansion of α and from b.
- Fast convolution algorithms
- Interval arithmetic to do all of the above calculations without floating point errors.
- The Savio supercomputer and ≈ 5 years of vCPU time.
- 10,000+ lines of code. (Half is various tests of correctness)

Thanks!