GENERIC GROUP PROPERTIES

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TOP. GROUPS: U.B. DARJI, A. KOCSIS, M. PÁLFY

FIXED COMMON UNIVERSE: $\mathbb{N}$

$\mathcal{G} = \{G \in \mathbb{N}^{\mathbb{N}\times\mathbb{N}} : G \text{ IS A GROUP OPERATION}\} \subseteq \mathbb{N}^{\mathbb{N}\times\mathbb{N}}$ POLISH

"THE SPACE OF GROUPS"

$S_\infty \curvearrowright \mathcal{G}$ CONTINUOUSLY: $\sigma . G =$ PUSH-FORWARD OF G ALONG σ

- ORBITS = ISOM. CLASSES
- THIS ACTION MAKES $\mathcal{G}$ NICE!
- BASIC OPEN SETS:

*	2	3	*	
*	*	*	*	
*	*	5	*	
		2		

<u>DEF:</u> A <u>GROUP PROPERTY</u> IS AN ISOM. INVARIANT SUBSET OF $\mathcal{G}$.

<u>Q:</u> WHAT GROUP PROPERTIES ARE GENERIC?

(A) COMEAGER ISOM. CLASS? NO!

(B) GENERIC SUBGROUPS?

(C) GEN. PROP. IN SUBSPACES?

(D) CONNECTIONS TO OTHER SETTINGS?

SEE ALSO: I. GOLDBRING, S. KUNNAWALKAM ELAYAVALLI, Y. LODHA

<u>DEF:</u> A FIN. GEN. GROUP G HAS <u>SOLVABLE WORD PROBLEM</u> IF $\exists$ TURING MACHINE THAT DECIDES FOR ANY WORD IN THE GENERATORS OF G WHETHER IT REPRESENTS THE IDENTITY IN G.

DEF: A GROUP G IS ALGEBRAICALLY CLOSED IF FOR EVERY FINITE SYSTEM Γ OF EQUATIONS AND INEQUATIONS WITH PARAMETERS FROM G WE HAVE:

$(\exists H \geq G \ \Gamma$ IS SOLVABLE IN $H) \Rightarrow (\Gamma$ IS SOLVABLE IN $G)$

EX:
$$\begin{cases} g_1 x_1 g_2^{-3} x_4 = 1 \\ g_2 x_2 = 1 \\ x_2^{-1} g_3 x_1 \neq 1 \end{cases}$$

THM: (GKL) ALG. CLOSEDNESS IS GENERIC.

COR: ALSO GENERIC:

- NOT LOC. FINITE
- NOT FIN. GEN.
- SIMPLE
- INNER ULTRAHOMOGENEOUS (HNN EXTENSIONS)

GENERIC SUBGROUPS (B)

DEF: THE GROUP H IS GENERICALLY EMBEDDABLE IF $\mathcal{E}_H = \{G \in \mathcal{G} : H$ EMBEDS INTO $G\}$ IS COMEAGER.

THM: (EGKKK) TFAE:

(1) H IS GEN. EMB.

(2) H CAN BE EMBEDDED INTO EVERY ALG. CLOSED GROUP

(3) H IS CTBL AND EACH OF ITS FIN. GEN. SUBGROUPS HAS SOLVABLE WORD PROBLEM

R: (2) $\Leftrightarrow$ (3) B.H. NEUMANN, H. SIMMONS, A. MACINTYRE (3 PAPERS!)

(2) $\Rightarrow$ (1) IS CLEAR FROM THE FACT THAT ALG. CLOSEDNESS IS GENERIC.

PROOF: (1) $\Rightarrow$ (2) (SKETCH)

SUFFICES: FOR H FIN. GEN. (BY HOMOGENEITY)

$\underline{\underline{G_0}}$ ALG. CLOSED, $\underline{\underline{H}}$ FIN. GEN. GEN. EMB.

$\mathcal{E}_H$ IS COMEAGER AND F_σ $\Rightarrow$ $\mathrm{int}(\mathcal{E}_H) \neq \emptyset$

<u>FACT:</u> THE ISOM. CLASS OF G_0 IS DENSE IN $\mathcal{G}$

$\Rightarrow$ ✓ $\square$

ISOM. CLASSES (A)

<u>THM:</u> (GKL) (0-1 LAW) ANY BAIRE MEASURABLE GROUP PROP. IS EITHER MEAGER OR COMEAGER.

<u>THM:</u> (GKL) EVERY ISOM. CLASS IS MEAGER IN $\mathcal{G}$.

<u>PROOF:</u> (SKETCH)

SUPPOSE: ISOM. CLASS OF G_0 IS NONMEAGER.

$\Rightarrow$ ASSUME IT IS COMEAGER. FOR H FIN. GEN.:

- $H \hookrightarrow G_0 \Leftrightarrow H$ GEN. EMB. $\Leftrightarrow$ H HAS SOLV. WORD PROBLEM.
- G_0 IS ALG. CLOSED
- FACT: EVERY ALG. CLOSED GROUP CONTAINS A FIN. GEN. SUBGROUP WITH UNSOLVABLE WORD PROBLEM.

G_0 CONTAINS A FIN. GEN. SUBG. WITH UNSOLV. WORD PROBLEM

↯

SUBSPACES (C)

NOW: ABELIAN FOR MORE: GKL

<u>THM:</u> (EGKKK) $\exists$ COMEAGER ISOM. CLASS IN THE SUBSPACE $\mathcal{A}$ OF ABELIAN GROUPS:

$$A = \bigoplus_{i \in \mathbb{N}} (\mathbb{Q}/\mathbb{Z})$$

<u>PROOF:</u> (SKETCH)

AS A CTBL AB. GROUP, A IS CHARACTERIZED BY 3 PROP.:

- TORSION

- DIVISIBLE
- EVERY FINITE AB. GROUP EMBEDS INTO A.

WE PROVE 1 BY 1 THAT THESE ARE GENERIC. $\square$

R: A IS THE FRAÏSSÉ LIMIT OF ALL FINITE AB. GROUPS

THM: (DEKKP) IN THE SUBSPACE OF TORSION-FREE ABELIAN GROUPS, THE ISOM. CLASS OF $(\mathbb{Q}, +)$ IS COMEAGER.

OTHER SETTINGS (D)

FACT: ANY CTBL GROUP G IS A QUOTIENT OF F_∞.

$G \cong F_\infty / N$.

$\mathcal{G}' = \{N \in 2^{F_\infty} : N \triangleleft F_\infty\} \subseteq 2^{F_\infty}$ POLISH

THM: (R. CHEN AND TK) $\mathcal{G}$ AND $\mathcal{G}'$ ARE EQUIVALENT IN TERM OF GENERIC PROPERTIES.