Isometry groups of Polish ultrametric spaces

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DI ECCELLENZA
MUR

Isometry groups of Polish metric spaces

If X is a Polish metric space, then $Iso(X)$ —the group of isometries of X— with the topology of pointwise convergence is a Polish group.

Theorem (Gao, Kechris 2003)

The isometry groups of Polish metric spaces are, up to topological isomorphism, all Polish groups.

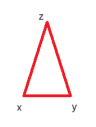
Similar characterisations are known for a few subclasses of Polish metric spaces:

- Locally compact Polish spaces and σ-compact Polish spaces correspond to subgroups of groups of the form $\prod_{n\in\mathbb{N}}(Sym(\mathbb{N}) \ltimes G_n^{\mathbb{N}})$ where G_n are locally compact Polish groups (Gao, Kechris 2003)
- Zero-dimensional locally compact Polish spaces correspond to closed subroups of $Sym(\mathbb{N})$ (Gao, Kechris 2003)
- Compact metric spaces correspond to compact Polish groups (Melleray 2008)
- Some classes of Polish ultrametric spaces (discussed later)

Definition (Krasner 1944)

A metric space (X, d) is **ultrametric** if d satisfies a stronger version of the triangle inequality:

$$d(x, z) = \max(d(x, y), d(y, z)), \quad \text{for all } x, y, z \in X$$



In fact, M. Krasner was a number theorist working on p-adic numbers, which are an example of ultrametric space.

Krasner's problem

Problem (Krasner 1956)

Provide a characterisation of the isometry groups of ultrametric spaces.

Note that Krasner's problem does not mention Polish spaces, though the main examples of interest for him (p-adic numbers) are indeed Polish.

A related problem to Krasner's was the following:

Problem (Pestov)

Provide a characterisation of all subgroups of the isometry groups of ultrametric spaces.

This was answered by A.Y. Lemin and Y.M. Smirnov (1986).

Krasner's problem for Polish ultrametric spaces

Problems (Gao, Kechris 2003)

- Characterise the isometry groups of Polish ultrametric spaces
- Characterise the isometry groups of Polish locally compact ultrametric spaces

Some starting points

- $Iso(X)$ is isomorphic to a closed subgroup of $Sym(\mathbb{N})$
- More specifically, isometry groups of Polish ultrametric spaces are isomorphic to automorphism groups of R-tree (Gao, Shao 2011)
- If X is not rigid, then $Iso(X)$ contains a non-trivial involution; so not all closed subgroups of $Sym(\mathbb{N})$ are isomorphic to some $Iso(X)$ (Gao, Kechris 2003)
- If $Iso(X)$ is simple, then $Iso(X) = \{id\}$, or $Iso(X) \simeq \mathbb{Z}_2$, or $Iso(X) \simeq Sym(\mathbb{N})$ (Malicki, Solecki 2009)
- If X is Heine-Borel, then $Iso(X)$ is the closure of an increasing union of compact subgroups (Gao, Kechris 2003), hence amenable

Krasner's problem for Polish ultrametric spaces

- If X is compact ultrametric (Feinberg 1974) or spherically complete ultrametric (Feinberg, Nosova 1980), then $Iso(X)$ is isomorphic to a generalised wreath product *à la Holland*

Definition (Malicki 2014)

A W-**space** is a Polish ultrametric space X such that:

- for any two $Iso(X)$-orbits O_1, O_2,

$$d(O_1, O_2) = \min\{d(x, y) \mid x \in O_1, y \in O_2\}$$

- X is *locally non-rigid*

- If X is a W-space, then $Iso(X)$ is a isomorphic to a variant of a generalised wreath product *à la Holland* (Malicki 2014, see also later)

Some unneeded philosophy

A *good* characterisation should:

- Make objects of independent interest —possibly coming from different areas— correspond
- Be potentially useful to attack some other questions
- ...

Of course, a good characterisation might not exist.

We try anyway to obtain a good characterisation.

L-trees

Let $L = (L, \leqslant_L)$ be a linear order.

Definition

A L-tree is a partial order $T = (T, \leqslant_T)$ together with a surjective map $lev_T : T \to L$ satisfying the following conditions, for every $t, t' \in T$:

1. $t \leqslant_T t' \Rightarrow lev_T(t) \leqslant_L lev_T(t')$
2.
3.
4.

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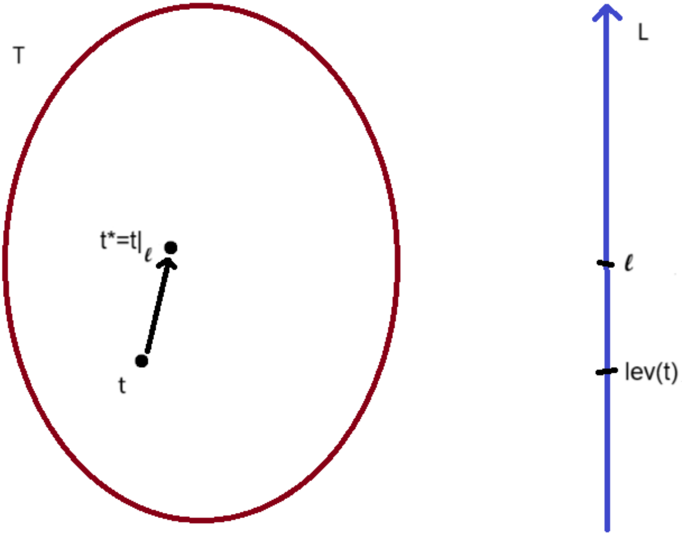
2. for every $\ell \geqslant_L lev_T(t)$ there exists a unique $t^* \in T$, denoted $t_{|_\ell}$, such that $t^* \geqslant_T t$ and $lev_T(t^*) = \ell$; in particular, $t_{|_{lev_T(t)}} = t$
3.
4.

L-trees



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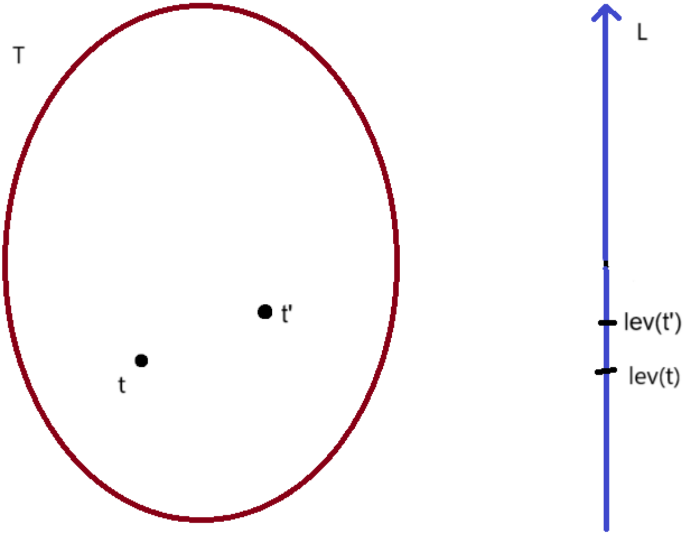
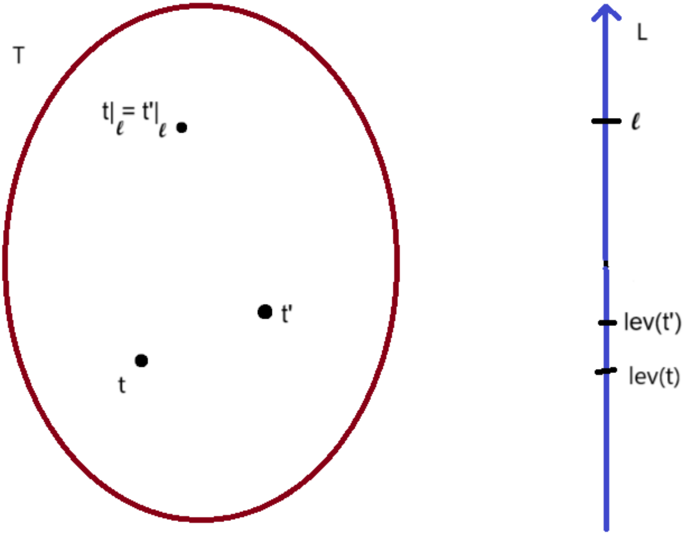
3. there is an upper bound for $\{t, t'\}$; equivalently, there exists $\ell \geqslant_L \max(lev_T(t), lev_T(t'))$ such that $t_{|_\ell} = t'_{|_\ell}$
4.


T
t'
t
L
lev(t')
lev(t)

L-trees



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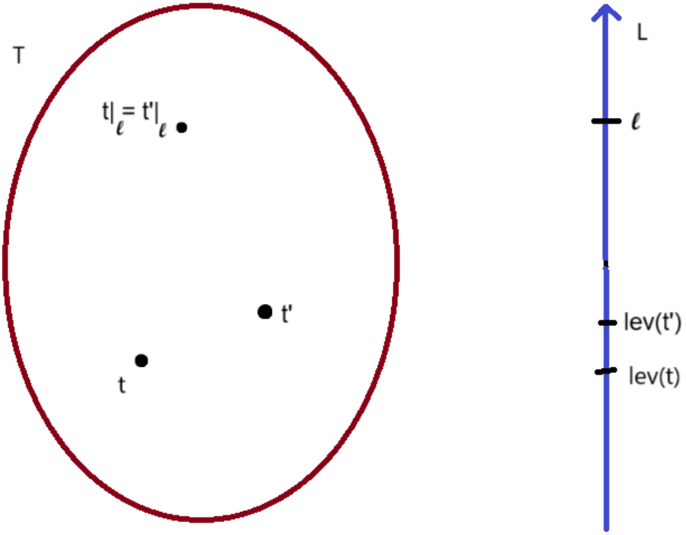
3. there is an upper bound for $\{t, t'\}$; equivalently, there exists $\ell \geqslant_L \max(lev_T(t), lev_T(t'))$ such that $t_{|_\ell} = t'_{|_\ell}$
4. if t, t' are $\leqslant_T$-incomparable, then $\{\ell \geqslant_L \max(lev_T(t), lev_T(t')) \mid t_{|_\ell} \neq t'_{|_\ell}\}$ has a maximum, denoted $split(t, t')$

L-trees



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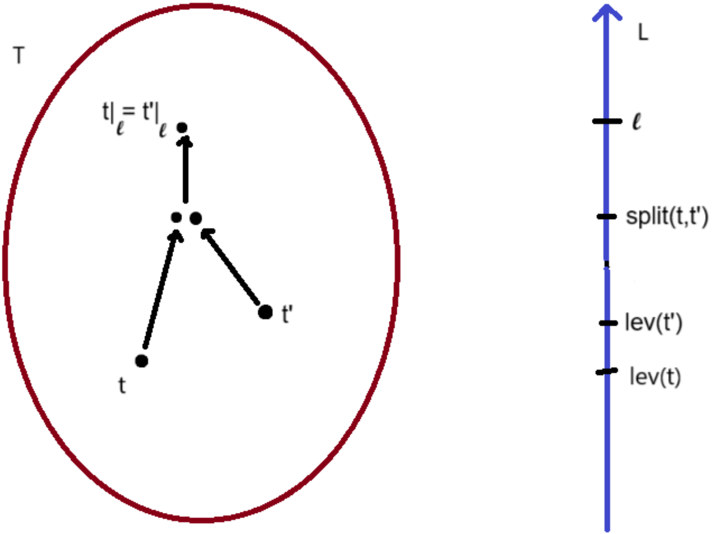


L-trees

Let T be a L-tree.

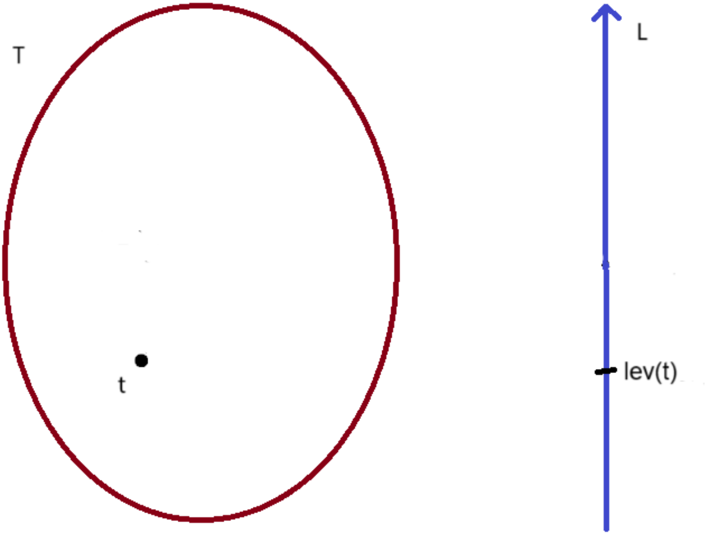
Definitions

$$[T] = \{b \in T^L \mid \forall \ell \in L \; lev_T b(\ell) = \ell \wedge \\ \wedge \forall \ell, \ell' \in L \; (\ell \leqslant_L \ell' \Rightarrow b(\ell) \leqslant_T b(\ell'))\}$$

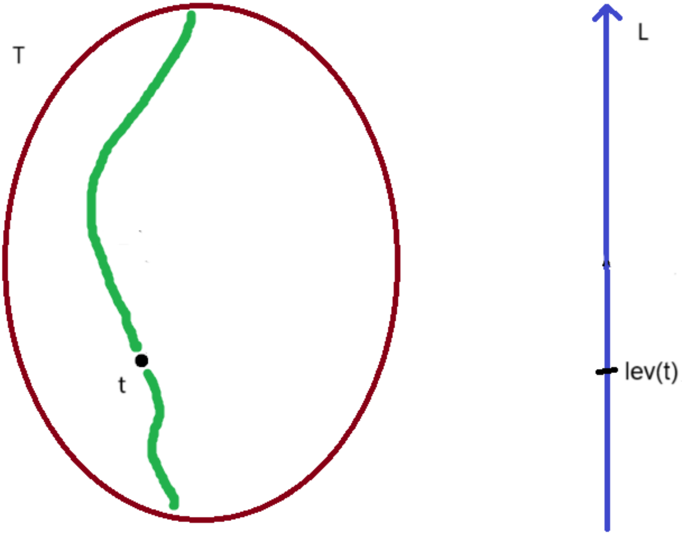
is the **body** of T

- the elements of $[T]$ are the **branches** of T
- T is **pruned** if for every $t \in T$ there exists $b \in [T]$ such that $b(lev_T(t)) = t$

L-trees



L-trees



Let T, S be L-trees.

- An **embedding** of T into S is an embedding $f : T \to S$ of the orders that respect levels; that is, an injective function such that:
  1. $\forall t \in T \ lev_T(t) = lev_S f(t)$
  2. $\forall t, t' \in T \ (t \leqslant_T t' \Leftrightarrow f(t) \leqslant_S f(t'))$
- An **isomorphism** from T to S is an isomorphism $f : T \to S$ of the orders that respect the levels; equivalently, a surjective embedding
- An isomorphism $f : T \to T$ is an **automorphisms** of T
- $Aut(T)$ is the group of automorphisms of T

If T is countable, then $Aut(T)$, equipped with the pointwise convergence topology, is a Polish group; indeed, it is topologically isomorphic to a closed subgroup of $Sym(\mathbb{N})$.

The following is a first answer to Gao and Kechris' problem (showing in particular that the answer to their two questions is the same):

Theorem

Let G be a topological group. Then the following are equivalent:

1. $G \cong Iso(X)$ for some Polish, ultrametric space X
2. $G \cong Iso(X)$ for some perfect, locally compact, Polish ultrametric space X
3. $G \cong Iso(X)$ for some uniformly discrete, Polish ultrametric space X
4. $G \cong Aut(T)$ for some countable, pruned L-tree T on some linear order L

The proof exploits the existence of suitable functors between relevant categories:

- for every countable $D \subseteq \mathbb{R}_0^+$, there exists a full embedding $\mathcal{F}$ from the category of Polish ultrametric spaces with distances in D (with isometric embedding) to the category of pruned $\mathbb{Q}$-trees (with embeddings)
- for every countable linear order L, there exists a full embedding $\mathcal{G}$ from the category of pruned L-trees (with embeddings) to the category of uniformly discrete, Polish ultrametric spaces (with isometric embeddings)
- there exist a full embedding $\mathcal{U}$ from the category of uniformly discrete, Polish ultrametric spaces to the category of perfect, locally compact, Polish ultrametric spaces (both with isometric embeddings)

These functors provide reductions between the corresponding relations of embeddability and isomorphism.

A corollary on Borel complexity

Corollary

1. The relation of isometry on the class of *perfect*, locally compact, Polish ultrametric spaces is $\mathit{Sym}(\mathbb{N})$-complete
2. The relation of isometric embeddability on the class of *perfect*, locally compact, Polish ultrametric spaces is invariantly universal, hence complete for analytic preorders

Proof. Argue as it was done in [Marcone, Motto Ros, C.; 2018] for other classes of spaces, using the functors of the previous page.

Generalised wreath products

There are several variants of generalised wreath products. Each one is based on two main ingredients:

- A **skeleton** (Δ, N), where Δ is a partial order and

$$\begin{array}{rccc} N: & \Delta & \to & Card \\ & \delta & \mapsto & N_\delta \end{array}$$

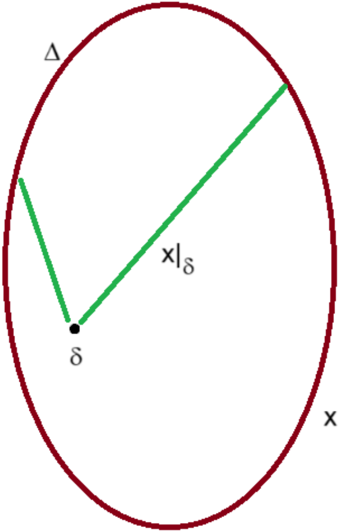
  with $N_\delta > 0$ for every δ.
  The skeleton is **countable** if so are Δ and all N_δ.
- A **domain** $S \subseteq \prod_{\delta\in\Delta} N_\delta$ such that for every $x \in S, \delta \in \Delta, \iota \in N_\delta$ there is some x^δ_ι satisfying

$$x^\delta_\iota(\delta) = \iota \quad \text{and} \quad \forall \gamma > \delta \; x^\delta_\iota(\gamma) = x(\gamma)$$

A convenient notation. If $x \in \prod_{\delta\in\Delta} N_\delta$ and $\delta \in \Delta$, denote

$$x_{|\delta} = x_{\restriction\{\gamma\in\Delta \mid \gamma\geqslant\delta\}} \in \prod_{\gamma\geqslant\delta} N_\gamma$$

and similarly for colouring of other partially ordered sets.


Δ
x|δ
δ
X

Given a skeleton (Δ, N) and a domain S, the associated **generalised wreath product** is an operator $Wr^S_{\delta\in\Delta}$ taking as input a sequence $(H_\delta)_{\delta\in\Delta}$ of *transitive* permutation groups $H_\delta \subseteq Sym(N_\delta)$ and yields a subgroup

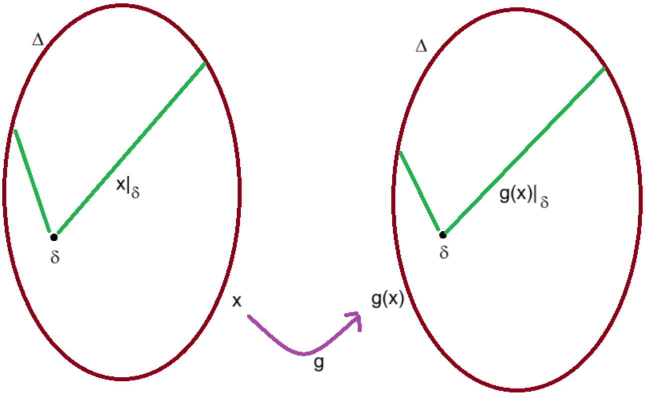
$$Wr^S_{\delta\in\Delta} H_\delta \subseteq Sym(S)$$

Definition

$Wr^S_{\delta\in\Delta} H_\delta$ is the subgroup consisting of all $g \in Sym(S)$ satisfying for every $x, y \in S, \delta \in \Delta$ the conditions:

- $x_{|\delta} = y_{|\delta} \Leftrightarrow (g(x))_{|\delta} = (g(y))_{|\delta}$
-

Generalised wreath products



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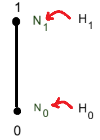
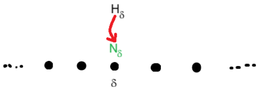
- the function $\iota \mapsto (g(x^\delta_\iota))(\delta)$ is a permutation of N_δ belonging to H_δ

(the second condition does not depend on the choice of x^δ_ι)

Examples



$$Wr_{\delta\in 2}^{N_0\times N_1} H_\delta = H_0 Wr H_1$$



If Δ is an antichain, then $Wr_{\delta\in\Delta}^{\prod_{\delta\in\Delta} N_\delta} H_\delta = \prod_{\delta\in\Delta} H_\delta$

A way to define a domain S is via supports.
If $x \in \prod_{\delta \in \Delta} N_\delta$, let

$$supp(x) = \{\delta \in \Delta \mid x(\delta) \neq 0\}$$

be the **support** of x.

If $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(\Delta)$ is such that $A \in \mathcal{A} \wedge (A \triangle A' \text{ finite}) \Rightarrow A' \in \mathcal{A}$, then

$$S^{\mathcal{A}} = \{x \in \prod_{\delta \in \Delta} N_\delta \mid supp(x) \in \mathcal{A}\}$$

is a domain. One can simplify notations by letting

$$Wr^{\mathcal{A}}_{\delta \in \Delta} H_\delta = Wr^{S^{\mathcal{A}}}_{\delta \in \Delta} H_\delta$$

Examples

Various choices of supports can be found in the literature, according to the problems at stake:

Examples

- (Hall 1962) Δ a linear order, $\mathcal{A} = Fin = \{A \subseteq \Delta \mid A \text{ is finite}\}$
- (Holland 1969) Δ arbitrary, $\mathcal{A} = Max = \{A \subseteq \Delta \mid A \text{ is anti-well-founded}\}$
- (Malicki 2014) Δ arbitrary, $\mathcal{A} = Wsp = \{A \in Max \mid$ every infinite decreasing sequence in A has no lower bound in $\Delta\}$
- Δ arbitrary, $\mathcal{A} = LF = \{A \subseteq \Delta \mid \forall \delta \in \Delta\ A \cap \{\gamma \in \Delta \mid \gamma \geqslant \delta\} \text{ is finite}\}$

Therefore:

- $Fin \subseteq LF \subseteq Wsp \subseteq Max$
- If Δ is a L-tree, then $LF = Wsp$

The topology of a generalised wreath product

Equip $\prod_{\delta \in \Delta} N_\delta$ with the topology τ_Δ generated by the sets

$$V_{x\gamma} = \{y \in \prod_{\delta \in \Delta} N_\delta \mid y_{|_\gamma} = x_{|_\gamma}\}, \quad \text{for } x \in \prod_{\delta \in \Delta} N_\delta, \gamma \in \Delta$$

When the skeleton (Δ, N) is countable:

- τ_Δ is generated by a complete ultrametric d_Δ
- Thus, if $S \subseteq \prod_{\delta \in \Delta} N_\delta$ is closed, it is completely metrisable
- If all $S_\delta = \{x_{|_\delta}\}_{x \in S}$ are countable, S is also separable

Then endow $Wr^S_{\delta \in \Delta} H_\delta$ with the topology of pointwise convergence with respect to the relativisation of τ_Δ to S.

The topology of a generalised wreath product

Theorem

Let (Δ, N) be a countable skeleton and $S \subseteq \prod_{\delta \in \Delta} N_\delta$ be a closed separable domain. If every $H_\delta \subseteq Sym(N_\delta)$ is a closed group, then $Wr^S_{\delta \in \Delta} H_\delta$ is a closed subgroup of $Iso(S, d_\Delta)$, hence a Polish group.

Examples

Let (Δ, N) be a countable skeleton. Then

- S^{LF}, S^{Wsp}, S^{Max} are closed; but
- in general only S^{LF} is separable —hence Polish

A metric space X is **homogeneous** if for every $x, y \in X$ there exists $\varphi \in Iso(X)$ such that $\varphi(x) = y$.

Definition

A L-tree T is **homogeneous** if for every $s, t \in T$ with $lev_T(s) = lev_T(t)$ there exists $\varphi \in Aut(T)$ such that $\varphi(s) = t$.

Theorem

Let G be a topological group. Then the following are equivalent:

1. $G \cong Iso(X)$ for some homogeneous, Polish ultrametric space X
2. $G \cong Aut(T)$ for some countable, pruned, homogeneous L-tree T
3. $G \cong Wr^{LF}_{\delta \in \Delta} Sym(N_\delta)$ when (Δ, N) is a countable skeleton with Δ a linear order

skip proof

The proof

The proof uses the following ingredients:

(1) ⇒ (2)

Functor $\mathcal{F}$ from Polish, ultrametric spaces to $\mathbb{Q}$-trees restricts to a functor that preserves homogeneity.

(2) ⇒ (3)

If T is a L-tree, let $t \sim t' \Leftrightarrow \exists\varphi \in Aut(T)\ \varphi(t) = t'$ and set $\Delta(T) = T/_\sim$, where

- $lev_{\Delta(T)}[t] = lev_T t$
- $[t] \leqslant_{\Delta(T)} [t'] \Leftrightarrow \exists s \in [t]\ \exists x' \in [t']\ s \leqslant_T s'$

If T is homogeneous, then $\Delta(T)$ is a linear order and

$$Aut(T) \cong Wr^{LF}_{\delta\in\Delta(T)} Sym(N_\delta)$$

where, for $\delta = [t]$,

$$N_\delta = \mathrm{card}(\{t' \in \delta \mid t' = t \vee split(t,t') = lev_T t\})$$

The proof

(3) $\Rightarrow$ (1)

Using the fact that Δ is a linear order, equip the domain S^{LF} of $Wr^{LF}_{\delta \in \Delta} Sym(N_\delta)$ with an ultrametric such that

$$Iso(S^{LF}) = Wr^{LF}_{\delta \in \Delta} Sym(N_\delta)$$

and S^{LF} turns out to be homogeneous

Discrete, homogeneous spaces

Theorem

Let G be a topological group. Then the following are equivalent:

1. $G \cong Iso(X)$ for some discrete, homogeneous, Polish ultrametric space X
2. $G \cong Aut(T)$ for some countable, homogeneous, pruned L-tree T where L has a minimum
3. $G \cong Wr^{Fin}_{\delta \in \Delta} Sym(N_\delta)$ for some countable, linear order Δ

Note. $Wr^{Fin}_{\delta \in \Delta} Sym(N_\delta)$ is Hall's generalised wreath product.

Orbits with realised distances

Homogeneous spaces are a particular case of spaces whose orbits have distances realised by points (a trivial particular case: there is just one orbit).

Recall:

Theorem (Malicki 2024)

If X is a Polish, ultrametric space such that

- the distance of any two orbits is realised by a pair of their points
- X is locally non-rigid

then $Iso(X)$ is of the form $Wr^{Wsp}_{\delta \in \Delta} Sym(N_\delta)$.

Indeed, when the space is not homogeneous, the fact that the distances of orbits are realised by points gives an important constraint on the isometry groups.

Orbits with realised distances

Definition

Let T be a L-tree.

- T is **special** if the quotient $\Delta(T) = T/\sim$ is a L-tree as well
- T satisfy condition $(*)$ if for every chain $C \subseteq T$ and every $t \in T$ such that $[t] \leqslant_{\Delta(T)} [C]$, there exists $t' \sim t$ such that $t' \leqslant_T C$

Theorem

Let G be a topological group. Then the following are equivalent:

1. $G \cong Iso(X)$ for X a Polish ultrametric space whose orbits have distances realised by points
2. $G = Aut(T)$ for T a countable, pruned, special L-tree T satisfying condition $(*)$
3. $G = Wr^{LF}_{\delta \in \Delta} Sym(N_\delta) = Wr^{Wsp}_{\delta \in \Delta} Sym(N_\delta)$ for Δ a countable, pruned L-tree

skip local domains

Even more generalised wreath products

To cover the general case, a further generalisation of wreath products is used. Recall the notation

$$S_\delta = \{x_{|_\delta}\}_{x \in S} \subseteq \prod_{\gamma \geqslant \delta} N_\gamma$$

for the restrictions of the elements in the domain of a wreath product. Recall also that a permutation $g : S \to S$ belongs to $Wr^S_{\delta \in \Delta} H_\delta$ iff:

1. $x_{|_\delta} = y_{|_\delta} \Leftrightarrow (g(x))_{|_\delta} = (g(y))_{|_\delta}$
2. the map $\iota \mapsto (g(x^\delta_\iota))(\delta)$ is a permutation of N_δ belonging to H_δ

Thus:

every $g \in Wr^S_{\delta \in \Delta} H_\delta$ induces permutations $g_\delta : S_\delta \to S_\delta$ that commute with restrictions: $(g_\delta(z))_{|_\gamma} = g_\gamma(z_{|_\gamma})$

$$\begin{tikzcd}
S_\gamma \arrow[r, "g_\gamma"] & S_\gamma \\
S_\delta \arrow[u, "restr^\gamma_\delta"] \arrow[r, "g_\delta"'] & S_\delta \arrow[u, "restr^\gamma_\delta"']
\end{tikzcd}$$

Even more generalised wreath products

Therefore, given a skeleton (Δ, N) and a *closed* domain $S \subseteq \prod_{\delta\in\Delta} N_\delta$, the following is an alternative presentation of $Wr^S_{\delta\in\Delta} H_\delta$, where $\mathsf{S} = \bigcup_{\delta\in\Delta} S_\delta$:

Alternative definition

The generalised wreath product $Wr^S_{\delta\in\Delta} H_\delta$ is the group of all permutations $g : \mathsf{S} \to \mathsf{S}$ such that for every $\delta \in \Delta, z \in S_\delta, \gamma \geqslant \delta$:

- $g(S_\delta) = S_\delta$
- $g(z_{|_\gamma}) = (g(z))_{|_\gamma}$ for every $\gamma \geqslant \delta$
- the map $\iota \mapsto (g(z_\iota))(\delta)$ is a permutation of N_δ belonging to H_δ, where $z_\iota(\gamma) = \begin{cases} z(\gamma) & \text{if } \gamma > \delta \\ \iota & \text{otherwise} \end{cases}$

Fix a skeleton (Δ, N). A **family of local domains** is a subset $\mathsf{S} \subseteq \bigcup_{\delta\in\Delta} \prod_{\gamma\geqslant\delta} N_\gamma$ such that, for every $\delta \in \Delta$:

- $\mathsf{S}_\delta = \mathsf{S} \cap \prod_{\gamma\geqslant\delta} N_\delta \neq \emptyset$
- $\gamma \geqslant \delta \Rightarrow \mathsf{S}_\gamma = \{z_{|_\gamma}\}_{z\in\mathsf{S}_\delta}$
- $\mathsf{S}_\delta = \{z_\iota \mid z \in \mathsf{S}_\delta, \iota \in N_\delta\}$

Definition

Let (Δ, N) be a skeleton. The **generalised wreath product over the family of local domains** S is the group $Wr^{\mathsf{S}}_{\delta\in\Delta} H_\delta$ of all permutations $g : \mathsf{S} \to \mathsf{S}$ satisfying for every $\delta \in \Delta, \gamma \geqslant \delta, z \in \mathsf{S}_\delta$:

1. $g(\mathsf{S}_\delta) = \mathsf{S}_\delta$
2. $g(z_{|_\gamma}) = (g(z))_{|_\gamma}$
3. the map $\iota \mapsto (g(z_\iota))(\delta)$ is a permutation of N_δ belonging to H_δ

Generalised wreath products over local domains

The group $Wr^{\mathsf{S}}_{\delta\in\Delta} H_\delta$ is endowed with the pointwise convergence topology, where S is given the discrete topology.

Definition

The family of local domains S is **full** if for every $\delta \in \Delta, z \in \mathsf{S}_\delta$ there exists $x \in \prod_{\gamma\in\Delta} N_\gamma$ such that $\forall\gamma \in \Delta\ x_{|_\gamma} \in \mathsf{S}_\gamma$.

Theorem

Let G be a topological group. Then the following are equivalent:

- $G \cong Wr^{S}_{\delta\in\Delta} H_\delta$ for some closed domain $S \subseteq \prod_{\delta\in\Delta} N_\delta$
- $G \cong Wr^{\mathsf{S}}_{\delta\in\Delta} H_\delta$ for some full family S of local domains

The restrictions

$$\begin{array}{ccc} \mathsf{S}_\delta & \to & \mathsf{S}_\gamma \\ z & \mapsto & z_{|_\gamma} \end{array}$$

can be replaced by other projections:

skip projective

Definition

Given a skeleton (Δ, N), a **system of projections** is a pair $(\mathbb{S}, \pi)$ such that:

- $\mathbb{S} \subseteq \bigcup_{\delta \in \Delta} \prod_{\gamma \geqslant \delta} N_\gamma$, satisfying the property $\mathbb{S}_\delta = \{z_\iota \mid z \in \mathbb{S}_\delta, \iota \in N_\delta\}$
- $\pi = \{\pi_{\delta\gamma} \mid \delta \leqslant \gamma\}$ is a family of surjective maps $\pi_{\delta\gamma} : \mathbb{S}_\delta \to \mathbb{S}_\gamma$, satisfying the properties
  - $\pi_{\delta\gamma}(z) = \pi_{\delta\gamma}(z') \Leftrightarrow z_{|_\gamma} = z'_{|_\gamma}$
  - $\pi_{\gamma\beta}\pi_{\delta\gamma} = \pi_{\delta\beta}$
    In particular, $\pi_{\delta\delta} = id$.

Projective wreath products

Definition

Let (Δ, N) be a skeleton and let $(\mathbb{S}, \pi)$ be a system of projections. The **projective wreath product**

$$Wr^{\mathbb{S}\pi}_{\delta\in\Delta} H_\delta$$

is the group of all permutations $g \in Sym(\mathbb{S})$ satisfying the properties:

- $g(\mathbb{S}_\delta) = \mathbb{S}_\delta$, for every $\delta \in \Delta$
- $g\pi_{\delta\gamma}(z) = \pi_{\delta\gamma}g(z)$, for every $\delta \leqslant \gamma, z \in \mathbb{S}_\delta$
- the map $\iota \mapsto (g(z_\iota))(\gamma)$ is a permutation of N_δ belonging to H_δ, for every $\delta \in \Delta, z \in \mathbb{S}_\delta$

The group $Wr^{\mathbb{S}\pi}_{\delta\in\Delta} H_\delta$ is again equipped with the pointwise convergence topology, where $\mathbb{S}$ is endowed with the discrete topology.

If $\mathbb{S}$ is countable and the groups H_δ are closed, $Wr^{\mathbb{S}\pi}_{\delta\in\Delta} H_\delta$ is a Polish group.

The general theorem

Theorem

Let G be a topological group. Then the following are equivalent:

1. $G \cong Iso(X)$ for some Polish ultrametric space X
2. $G \cong Iso(X)$ for some perfect, locally compact, Polish ultrametric space X
3. $G \cong Iso(X)$ for some discrete, Polish ultrametric space X
4. $G \cong Aut(T)$ for some countable, pruned L-tree T
5. $G \cong Wr^{\mathsf{S}}_{\delta \in \Delta} Sym(N_\delta)$ for some countable, pruned L-tree Δ and countable family S of local domains with $\mathsf{S} \subseteq \mathsf{S}^{Max}$
6. $G \cong Wr^{\mathbb{S}\pi}_{\delta \in \Delta} Sym(N_\delta)$ for some countable, pruned L-tree Δ and some arbitrary system of projections $(\mathbb{S}, \pi)$ with $\mathbb{S} \subseteq \mathbb{S}^{Max}$

Does this provide a *good* characterisation?

Due to its construction, the proposed characterisation has some desirable features.

(∗) It is versatile: given a subclass of ultrametric Polish space, one can often explicitely find the corresponding class of automorphisms of L-trees or of wreath products —some examples have been given in the talk

(∗∗) It allows explicit computations

($*$) Some correspondences

Isometry groups of:	**Automorphism groups**	**Wreath products**
all	L-trees	$Wr^{\mathsf{S}}_{\delta\in\Delta} Sym(N_\delta)$ $\mathsf{S} \subseteq \mathsf{S}^{Max}$
locally compact	L-trees	$Wr^{\mathsf{S}}_{\delta\in\Delta} Sym(N_\delta)$ $\mathsf{S} \subseteq \mathsf{S}^{Max}$
discrete homogeneous	homogeneous L-tree L with a minimum	$Wr^{Fin}_{\delta\in\Delta} Sym(N_\delta)$ Δ linear
homogeneous	homogeneous L-trees	$Wr^{LF}_{\delta\in\Delta} Sym(N_\delta)$ Δ linear
orbits with realised distances	L-trees with property ($*$)	$W^{LF}_{\delta\in\Delta} Sym(N_\delta)$ Δ an L-tree

(**) An example: Polish ultrametric Urysohn spaces

Definition (Gao, Shao 2011)

Let R be a countable subset of $\mathbb{R}^+$. A *R-ultrametric Urysohn space* is an ultrametric space X such that:

- the distances between distinct elements of X belong to R (that is, X is a *R-ultrametric space*)
- for every finite R-ultrametric space B, every $A \subseteq B$, every isometric embedding $\varphi : A \to X$ there is an isometric embedding $\varphi^* : B \to X$ extending A

Theorem

Let G be a topological group. The following are equivalent.

- $G \cong Iso(U)$ for some Polish ultrametric Urysohn space U
- $G \cong Wr^{LF}_{\delta \in \Delta} Sym(N_\delta)$ where Δ is linear and $N_\delta \in \{1, \omega\}$ for every $\delta \in \Delta$
- $G \cong Wr^{LF}_{\delta \in \Delta} Sym(N_\delta)$ where Δ is linear and $N_\delta = \omega$ for every $\delta \in \Delta$

Further developments

1. Use the characterisations to compute explicitly the isometry group of other Polish ultrametric spaces
2. Find similar characterisations for the isometry groups of other classes of Polish ultrametric spaces
3. Find a way to recognise when two generalised wreath products are isomorphic
4. Compare the various versions of generalised wreath products
5. The description of the isometry group of a Polish ultrametric space X using wreath products might help to answer questions like:
   - When conjugacy on $Iso(X)$ is $Sym(\mathbb{N})$-complete?
   - When $Iso(X)$ is amenable?
   - ...
6. Note however that the most general formulation of Krasner's problem is still open:

Problem (Krasner 1956)

Characterise (at least, algebraically) the isometry groups of ultrametric spaces.