The Radon-Nikodym topography of measure-class-preserving equivalence relations

Anush Tserunyan (McGill University)

Joint with Robin Tucker-Drob; Ruiyuan Chen & Grigory Terlov

Measure-class-preserving equivalence relations

— Let R be a ctbl Borel equivalence relation (cBer) on a standard probabity space (X, μ).

— Feldman-Moore: all cBers are orbit equivalence relations Borel actions $\Gamma \curvearrowright X$ of ctbl groups, as well as connectedness relations R_G of locally ctbl Borel graphs.

— A cBer R on (X, μ) is

- probability-measure-preserving (pmp) if it is induced by a Borel action of a ctbl group $\Gamma \curvearrowright X$ which preserves the measure μ.
- measure-class-preserving (mcp) if ——— preserves the class of μ, i.e. preserves μ-null sets.

On the level of points: mcp $\Leftrightarrow$ there is an essentially unique Borel $w : R \to \mathbb{R}_{>0}$, $(x,y) \mapsto w^y(x) = \frac{\text{weight}(x)}{\text{weight}(y)}$, called the Radon-Nikodym cocycle of R wrt μ, such that

(i) w is a cocycle: $w^z(y) \cdot w^y(x) = w^z(x)$ for all R-related $x, y, z \in X$,

(ii) w satisfies mass transport: for each Borel $f : R \to [0, \infty]$

$$\int \sum_{y \in [x]_R} f(x,y) \, d\mu(x) = \int \sum_{y \in [x]_R} f(y,x) \cdot w^x(y) \, d\mu(x).$$

— pmp $\Leftrightarrow$ $w \equiv 1$.

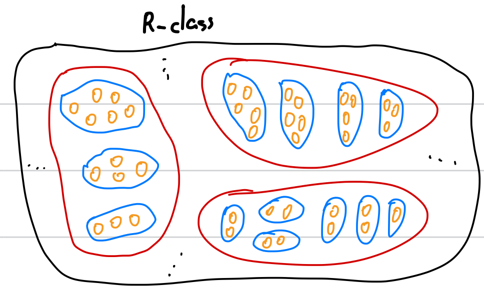
Examples of mcp (non-pmp) cBers.

ⓐ Poisson boundaries and other boundary actions, like those of hyperbolic groups.

ⓑ Cross-section equivalence relations of pmp actions of nonunimodular loc. compact 2nd ctbl groups.

ⓒ Cluster equivalence relations of invariant percolations on nonunimodular graphs.

Amenable cBers.



— The simplest yet/hence most important class of cBers are hyperfinite ones: $R = \biguplus_{n\in\mathbb{N}} R_n$, where each R_n is a finite Borel equivalence relation.

— We work in the context of measure, where we allow discarding R-invariant null sets. In this context, hyperfinite = amenable (by the Connes-Feldman-Weiss theorem):

— An mcp cBer R on (X,μ) is called μ-amenable if it admits Reiter functions, i.e. Borel functions $m_n : R \to [0,\infty)$ such that

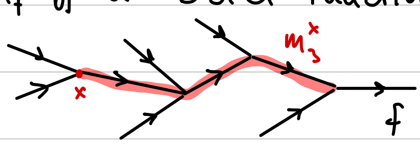
(i) $\|m_n^x\|_1 = 1$ for all $n \in \mathbb{N}$ and $x \in X$;

(ii) $\|m_n^x - m_n^y\| \to 0$ as $n \to \infty$ for all x, y in an R-invariant conull set.

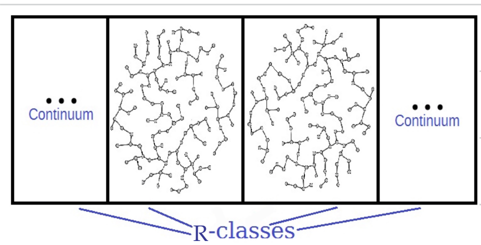
Examples.

(a) Borel actions of amenable groups yield amenable cBers.

(b) The orbit eq. rel. R_f of a Borel function $f : X \to X$ are amenable:

R_f-class: 

Treeable cBers.



— A graphing of a cBer R is a Borel graph G whose connected components are exactly the R-classes.

— A cBer is treeable if it admits an acyclic graphing, called a treeing. These form a delicate class, analogous to free groups.

Examples. (a) Hyperfinite cBers are treeable because they are induced by Borel actions of $\mathbb{Z}$ (Weiss, Slaman-Steel).

(b) Free Borel actions of the following groups give treeable cBers:

(i) Free groups

Treeable cBer — by Tasmin Chu

(ii) virtually free groups (Jackson–Kechris–Louveau)
(iii) surface groups, after discarding a null set (Conley–Gaboriau–Marks–Tucker-Drob)

pmp CBERs: detecting amenability from geometry.

Which pmp CBERs are amenable?

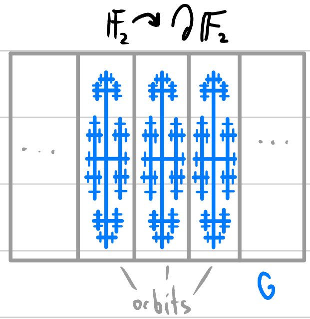
Amenability via group actions. Let R_Γ be the orbit eq. rel. of a free pmp action $\Gamma \curvearrowright (X,\mu)$. Then R_Γ is amenable $\Leftrightarrow$ Γ is amenable.

Amenability via the geometry of treeings. Because amenable CBERs are treeable a.e., it makes sense to ask: which acyclic Borel graphs on (X,μ) are amenable (i.e. have amenable connectedness relations)?

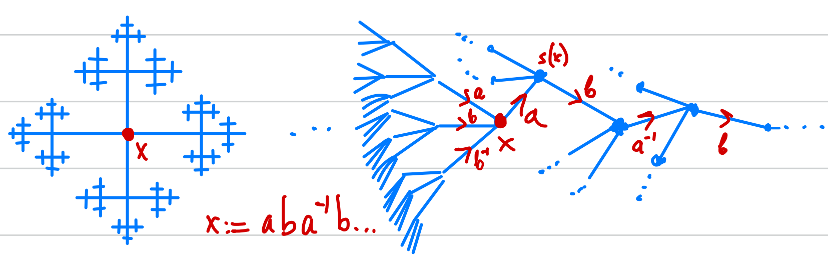
Adams Dichotomy for pmp. Let T be an acyclic pmp graph on (X,μ). Then:

1. R_T is amenable $\Leftrightarrow$ a.e. T-component has ≤ 2 ends.
2. R_T is nowhere amenable $\Leftrightarrow$ a.e. T-component has perfectly many ends (i.e. the end space is perfect Polish).

Counterexample in the mcp setting.



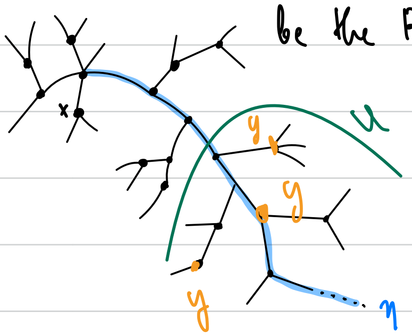
- Let $\mathbb{F}_2 := \langle a, b \rangle$ and treat the boundary $\partial\mathbb{F}_2$ as the subspace of $\{a^{\pm1}, b^{\pm1}\}^{\mathbb{N}}$ of infinite reduced words, so $\mathbb{F}_2 \curvearrowright \partial\mathbb{F}_2$ by concatenation-and-cancellation.
- Equip $\partial\mathbb{F}_2$ with the "uniform" measure μ, i.e. for a finite reduced word $w_0 w_1 \dots w_n$
$$\mu([w_0 w_1 \dots w_n]) := \frac{1}{4}\cdot\left(\frac{1}{3}\right)^n.$$
- Then $\mathbb{F}_2 \curvearrowright \partial\mathbb{F}_2$ is free a.e. (off of a ctbl set), so the Schreier graph is a forest of 4-regular trees.
- Despite this and the nonamenability of $\mathbb{F}_2$, the orbit eq. rel. $R_{\mathbb{F}_2}$ is amenable (actually Borel hyperfinite) since $R_{\mathbb{F}_2} = R_s$ for the left-shift map $s: \partial\mathbb{F}_2 \to \partial\mathbb{F}_2$.



mcp setting: detecting amenability from topography. The main realization is that not all ends have equal weight (pun intended). Indeed, in the last example, the Radon-Nikodym cocycle w grows (in powers of 3) to ∞ towards the forward end of the shift, so it decays (in powers of $\frac{1}{3}$) to 0 towards back ends of the shift.

There are many options for a definition of the special ends, but the following is the only one that works and generalizes to non-acyclic graphs:

Def (Ts.-Tucker-Drob, Chen-Terlov-Ts.) Let G be a locally ctbl mcp graph on (X,μ) and let $w: R_G \to \mathbb{R}_{>0}$ be the Radon-Nikodym cocycle of (R_G,μ). Call an end η of G *w-vanishing* if



$$\lim_{y\to\eta} w^x(y) = 0$$

where x is any point in the connected component of η ($\lim = 0$ is independent of x), i.e. $\forall \varepsilon > 0$ $\exists$ neighbourhood U of η such that $w^x(y) < \varepsilon$ for all $y \in U$.

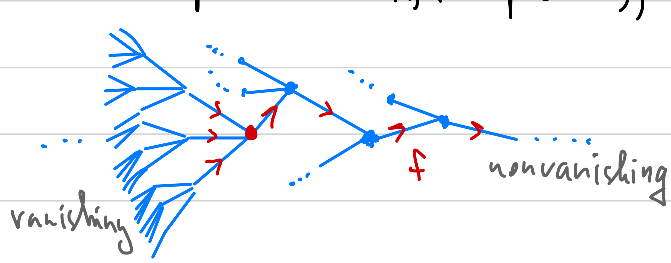
Thus, η is *w-nonvanishing* if $\exists \varepsilon > 0$ $\forall$ neighbourhood U of η such that $w^x(y) \geq \varepsilon$ for some $y \in U$.

Generalization of Adams dichotomy (Ts.-Tucker-Drob). Let T be an mcp acyclic Borel graph on (X,μ). Let $w: R_T \to \mathbb{R}_{>0}$ be the Radon-Nikodym cocycle of (R_T,μ). Then:

1. R_T is amenable $\iff$ a.e. T-component has ≤ 2 w-nonvanishing ends.
2. R_T is nowhere amenable $\iff$ a.e. T-component has perfectly many w-nonvanishing ends.

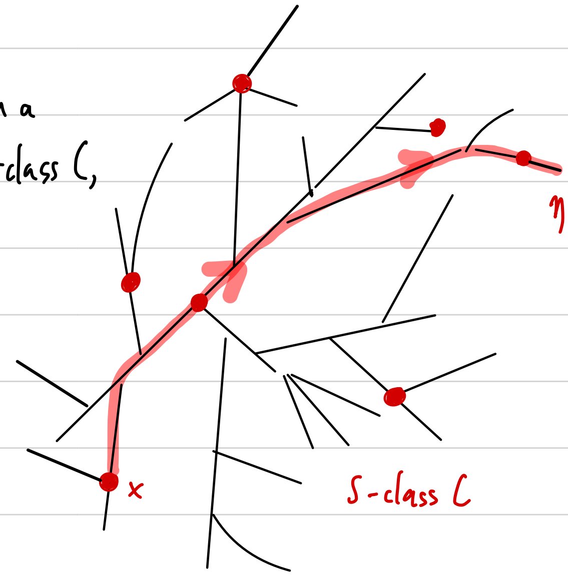
This mainly relies on analyzing ctbl-to-one Borel functions:

Main Lemma. Let $f: X \to X$ be an acyclic ctbl-to-one Borel map such that R_f is mcp on (X,μ), nowhere smooth, and f is not essentially two-ended. Then, for a.e. f-orbit, the f-forward end is nonvanishing while all back-ends are vanishing.



Applications (Ŏ – Tucker-Drob)

We consider amenable subrelations of treeable equivalence relations and make the Adams-Lyons end-selection theorem completely transparent:



Theorem (selecting nonvanishing ends). Let R be an mcp cBer on (X,μ) with a treeing T, and let $S \subseteq R$ be an amenable subrelation. Then for a.e. S-class C, there are ≤ 2 ends of T that are nonvanishing along C.
(Hence these are exactly the ends that C selects!)

This has many applications. The first two generalize theorems of Lewis Bowen from pmp to mcp:

Corollary (constancy of end-selection). Let R be an mcp cBer on (X,μ) with a treeing T.
If $S_1 \leq S_2$ are nowhere smooth amenable subrelations of R, then they "select" the same ends of T a.e., i.e. the Borel maximal T-end-selections of S_1 and S_2 coincide a.e.

Corollary (unique extendability). Let R be treeable mcp cBer on (X,μ). Then each nowhere smooth amenable subrelation $S \subseteq R$ admits an essentially unique maximal amenable extension $S \subseteq \bar{S} \leq R$.

Antitreeability criterion. Let R be an mcp cBer on (X,μ). If R admits amenable subrelations S_1 and S_2 such that $S_1 \cap S_2$ is nowhere smooth and $S_1 \vee S_2$ is nowhere amenable, then R is nowhere treeable. Call (S_1, S_2) an antitriable configuration.

Examples. (a) Take an mcp action $\mathbb{F}_2 \times \mathbb{Z} \curvearrowright (X,\mu)$ so that the orbit eq. rel. $R_{\mathbb{F}_2 \times \mathbb{Z}}$ is nowhere amenable and the subrelation $R_{1 \times \mathbb{Z}}$ is nowhere smooth. Then $R_{\mathbb{F}_2 \times \mathbb{Z}}$ is nowhere treeable.
(b) Antitreeability results for products of (nonunimodular) locally compact 2^{nd} ctbl groups.

Day–von Neumann style results (Chen–Terlov–Ţ)

Recall the Day–von Neumann question: does every nonamenable cfbl group Γ contain $\mathbb{F}_2$?

Ol'shanski (1980). No!

Gaborian–Lyons (2009). Measurably yes! If Γ is nonamenable, then the orbit eq. rel. E_Γ of the Bernoulli shift $\Gamma \curvearrowright ([0,1]^\Gamma, \lambda^\Gamma)$ contains the orbit eq. rel. $E_{\mathbb{F}_2}$ of a free ergodic action of $\mathbb{F}_2$.

A key ingredient of a proof of the Gaborian–Lyons theorem is:

Theorem (Gaborian 2000 + Ghys 1995). Let G be a loc. finite ergodic pmp graph on (X,μ). If a.e. G-component has >2 ends then G is nowhere amenable; in fact, R_G contains a nowhere amenable ergodic subforest.

Generalization of the Gaborian–Ghys Theorem (Chen–Terlov–Ţ). Let G be a loc. finite ergodic mcp graph on (X,μ). If a.e. G-component has >2 nonvanishing ends, then G is nowhere amenable; in fact, G contains a nowhere amenable ergodic subforest T (i.e. a.e. T-component has >2 nonvanishing ends).

Remark. We can get an ergodic subforest of G, and not just of R_G, due to the existence of ergodic hyperfinite subgraphs (Tucker-Drob for pmp, Ţ for mcp).

— The Gaborian–Ghys proof uses a construction from geometric group theory (by Stallings) to conclude that R_G is a free product $R_1 * R_2$, and then the theory of cost gives nonamenability and a witnessing subforest of R_G.
— But the theory of cost only works in the pmp setting, so we couldn't adapt this proof.
— Instead, we came up with a simple "weighted" cycle-cutting algorithm to get the subforest. The main work goes into proving that this subforest has >2 nonvanishing ends.