Hyper-u-amenability and hyperfiniteness of treeable equivalence relations

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The difficulties that arise while attempting to expand our current understanding of hyperfinite countable Borel equivalence relations often boil down to two main themes of inquiry:

- The relationship in the Borel context between **hyperfiniteness** and **amenability**.
- The difference between the **purely Borel** setting and the **measure-theoretic** one.

The project I'll present today addresses some specific instances of these themes, in the framework of **treeable** equivalence relations.

Hyperfiniteness and amenability

- A **countable Borel equivalence relation** (**CBER**) E on a standard Borel space X is an equivalence relation whose graph $E \subseteq X$ is Borel and whose classes are countable.

Feldmann-Moore, 1977: Every CBER E is equal to the orbit equivalence relation E_G^X of a Borel action $G \curvearrowright X$, for G countable.

- A CBER E on X is **hyperfinite** if there is an increasing countable sequence $E_0 \subseteq E_1 \subseteq \ldots$ of **finite** CBERs such that $E = \bigcup_{n \in \mathbb{N}} E_n$.
- A CBER E on X is **amenable** if it admits **Reiter functions**, namely a sequence of Borel maps $\lambda_n : E \to \mathbb{R}^+$ such that

  1. $\lambda_{n,x} \in \ell^1([x]_E)$ and $\|\lambda_{n,x}\|_1 = 1$,
  2. $\|\lambda_{n,x} - \lambda_{n,y}\|_1 \to 0$ as $n \to \infty$, for xEy (pointwise approximate invariance).

Amenability $\Longleftarrow$ **Hyperfiniteness**

Amenability $\Longleftarrow$ **Hyperfiniteness**

$\Uparrow$

$E = E_G^X$
for G amenable

$\swarrow\!\!\!\swarrow$

Weiss, 1984; Slaman-Steel, 1988: A CBER E is hyperfinite if and only if $E = E_{\mathbb{Z}}^X$, for some Borel action $\mathbb{Z} \curvearrowright X$.

Amenability $\Longleftarrow$ **Hyperfiniteness** (⇢ ?)

$\Uparrow$

$E = E_G^X$
for G amenable

$\Swarrow$ (⇢ ?)

Question (Weiss, 1984): Let G be a countable amenable group and let $G \curvearrowright X$ be a Borel action. Is E_G^X hyperfinite?

Question (Jackson-Kechris-Louveau, 2002): Are amenable CBERs hyperfinite?

Connes-Feldmann-Weiss, 1981

Measure-amenability $\Longleftrightarrow$ **Measure-hyperfiniteness**

Free continuous actions on σ-compact spaces $\Downarrow$ $\Uparrow$ $\qquad$ $\Uparrow$ $\downarrow$?

$\dashrightarrow$?

Amenability $\Longleftarrow$ **Hyperfiniteness**

$\Uparrow$

$E = E_G^X$ for G amenable $\qquad$ $\Leftarrow$ $\dashrightarrow$?

Let μ be a Borel probability measure on X. Then E is **μ-hyperfinite/amenable** if it is hyperfinite/amenable on a μ-measure 1 subset of X.

The CBER E is **measure-hyperfinite/amenable** if it is μ-hyperfinite/amenable for every Borel probability measure μ on X.

Question: Does measure-hyperfiniteness imply hyperfiniteness?

Treeable relations

A CBER $E \subseteq X^2$ is **treeable** if there is an acyclic Borel graph $\mathcal{G} = (X, R)$ such that $E = E_{\mathcal{G}}$.

Hyperfinite CBERs are always **treeable**!

Example: Orbit equivalence relations of free Borel actions of free groups $F_n \curvearrowright X$. (even virtually free)

Treeable equivalence relations are not hyperfinite in general (e.g. the free part of the shift action $F_2 \curvearrowright 2^{F_2}$).

A tree with a cycle.

Treeable amenable relations

Jackson-Kechris-Louveau, 2002: Are amenable **treeable** CBERs hyperfinite?

Theorem (Naryshkin-V., 2025)

Let E be a hyper-u-amenable, treeable CBER. Then E is hyperfinite.

Corollary (Naryshkin-V., 2025)

Let F be a virtually free group and let $F \curvearrowright X$ be a free continuous action on a σ-compact Polish space. If E_F^X is measure-hyperfinite (e.g. amenable), then it is hyperfinite.

Corollary (Naryshkin-V., 2025)

Let G be a countable amenable group and let $G \curvearrowright X$ be a Borel action. If E_G^X is **treeable**, then it is hyperfinite.

Borel extended metric spaces

A **Borel extended metric space** is a pair (X, ρ) where X is a standard Borel space and ρ is a Borel metric that can also take value ∞.

$$E_\rho := \{(x, y) \in X^2 : \rho(x, y) < \infty\}$$

Example: A Borel graph $\mathcal{G} = (X, R)$ with the shortest path metric $\rho_\mathcal{G}$. In this case $E_{\rho_\mathcal{G}} = E_\mathcal{G}$, where $E_\mathcal{G}$ is given by the connected components of $\mathcal{G}$.

Example: Let E be a CBER and let $G \curvearrowright X$ be a Borel action such that $E = E_G^X$. Given $S \subseteq G$ symmetric set of generators of G, the **Schreier graph** $\mathrm{Sch}(S, X) := (X, R)$ is

$$xRy \iff \exists g \in S \setminus \{e\} \text{ s.t. } gx = y.$$

and $E = E_G^X = E_{\mathrm{Sch}(S,X)}$.

Borel asymptotic dimension

Definition (Conley-Jackson-Marks-Seward-Tucker-Drob, 2023)

Let (X, ρ) be an extended metric space. The **Borel asymptotic dimension** of (X, ρ), denoted $\mathrm{asdim}_B(X, \rho)$, is the smallest $d \in \mathbb{N}$ such that for every $r > 0$ there is a ρ-uniformly bounded Borel equivalence relation E such that $B_\rho(x, r)$ meets at most $d + 1$ E-classes, and it is ∞ if no such d exists.

Theorem (Conley-Jackson-Marks-Seward-Tucker-Drob, 2023)

1. If (X, ρ) is proper and $\mathrm{asdim}_B(X, \rho) < \infty$, then E_ρ is hyperfinite.
2. If $(\rho_n)_{n\in\mathbb{N}}$ are proper extended Borel metrics on X such that $\rho_n \leq \rho_{n+1}$, that $E = \bigcup_{n\in\mathbb{N}} E_{\rho_n}$ and that $\mathrm{asdim}_B(X, \rho_n) < \infty$, then E is hyperfinite.

Some results using Borel asymptotic dimension

Theorem (Conley-Jackson-Marks-Seward-Tucker-Drob, 2023)

Let G be the lamplighter group, a group of uniform local polynomial volume-growth, or a virtually solvable group with finite Prüfer rank, and let $G \curvearrowright X$ be a free Borel action. Then $\dim_B(G \curvearrowright X) < \infty$, and in particular E_G^X is hyperfinite.

Theorem (Naryshkin-V., 2025)

Let G a finitely generated hyperbolic group and let ∂G its Gromov boundary. Then $\dim_B(G \curvearrowright \partial G) < \infty$ (in particular $E_G^{\partial G}$ is hyperfinite).

Remark: Marquis and Sabok already proved that $E_G^{\partial G}$ is hyperfinite (2020).

Theorem (Iyer-Shinko , 2024)

If G is locally of finite asymptotic dimension, then the generic continuous G-action on the Cantor space is hyperfinite.

U-amenability and hyper-u-amenability

Definition (Naryshkin-V., 2025)

Let (X, ρ) be an extended metric space and let $E \subseteq X^2$ be a CBER. E is **u-amenable with respect to ρ** if $E \subseteq E_\rho$ and if there are Borel maps

$$\lambda_n : E \to \mathbb{R}_{\geq 0}, \quad n \in \mathbb{N}$$

such that, with $\lambda_{n,x}(\,\cdot\,) := \lambda_n(x, \,\cdot\,)$,

1. $\lambda_{n,x} \in \ell^1([x]_E)$ and $\|\lambda_{n,x}\|_1 = 1$,

2. $\sup_{\{(x,y)\in E:\rho(x,y)<r\}} \|\lambda_{n,x} - \lambda_{n,y}\|_1 \to 0$ as $n \to \infty$, for every $r > 0$ (uniform approximate invariance).

A Borel graph $\mathcal{G}$ is u-amenable if $E_{\mathcal{G}}$ is **u-amenable** with respect to $\rho_{\mathcal{G}}$.

Definition (Naryshkin-V., 2025)

A CBER $E \subseteq X^2$ is **hyper-u-amenable** if it admits a graphing which is (or, equivalently, if all its graphings are) union of an increasing countable sequence of Borel u-amenable graphs.

U-amenable equivalence relations

Proposition

Let G be a finitely generated amenable group, let $G \curvearrowright X$ be a Borel action and let $\mathcal{G}$ be the corresponding Schreier graph. Then E_G^X is u-amenable with respect to $\rho_{\mathcal{G}}$.

In particular, E_G^X is hyper-u-amenable for every Borel $G \curvearrowright X$ with G amenable.

Proof.

Let $(F_n)_{n\in\mathbb{N}}$ be a Følner sequence for G and set

$$\lambda_n : E_{\mathcal{G}} \to \mathbb{R}_{\geq 0}$$
$$(x, y) \mapsto \frac{1}{|F_n|} |\{g \in F_n : gx = y\}|$$

U-amenable equivalence relations II

Proposition

Let G be a finitely generated group, let $G \curvearrowright X$ be a continuous free action on a compact Polish space, and let $\mathscr{G}$ be the corresponding Schreier graph. If E_G^X is amenable, then it is u-amenable with respect to $\rho_{\mathscr{G}}$.

In particular, E_G^X is hyper-u-amenable for every continuous free action $G \curvearrowright X$ on a σ-compact Polish space such that E_G^X is amenable.

Idea of proof.

In the case of free actions, (measure-)amenability is equivalent to *topological* amenability of the action, and u-amenability follows from pointwise approximate invariance by a compactness argument.

Hyper-u-amenable equivalence relations

Proposition

The following CBERs are hyper-u-amenable:

1. E_G^X whenever $G \curvearrowright X$ is a Borel action of an amenable group.
2. E_G^X whenever it is measure-hyperfinite and $G \curvearrowright X$ is a continuous free action on a σ-compact Polish space (freeness can be weakened to requiring amenable stabilizers)
3. Every amenable **Borel bounded** CBER.

Definition (Boykin-Jackson, 2007)

A CBER E on X is **Borel bounded** if for every Borel function $\varphi: X \to \mathbb{N}^{\mathbb{N}}$ there is a Borel function $\psi: X \to \mathbb{N}^{\mathbb{N}}$ such that $\varphi(x) \leq^* \psi(x)$ for all $x \in X$, and such that $\psi(x) =^* \psi(y)$ whenever xEy.

(Hyper-)u-amenability and hyperfiniteness

Theorem (Naryshkin-V., 2025)

Let $\mathscr{G} = (X, R)$ be a Borel acyclic graph with bounded degree. If $\mathscr{G}$ is u-amenable, then $\mathrm{asdim}_B(X, \rho_{\mathscr{G}}) < \infty$. In particular $E_{\mathscr{G}}$ is hyperfinite.

Corollary

Let E be a hyper-u-amenable, treeable CBER. Then E is hyperfinite.

Corollary

The following CBERs are hyperfinite:

1. E_G^X whenever it is treeable and $G \curvearrowright X$ is a Borel action of an amenable group.
2. E_G^X whenever it is measure-hyperfinite, treeable and $G \curvearrowright X$ is a continuous action with amenable stabilizer (e.g. a free action) on a σ-compact Polish space.
3. Every amenable, treeable, Borel bounded CBER.

Borel orientations

Let $\mathscr{G} = (X, R)$ be a Borel graph. A **Borel orientation** is a Borel subset $R_\rightarrow \subseteq R$ such that $R_\rightarrow \cap R_\rightarrow^{-1} = \varnothing$ and $R_\rightarrow \sqcup R_\rightarrow^{-1} = R$

Proposition

Let $\mathscr{G} = (X, R)$ be a Borel graph with bounded degree. If $\mathscr{G}$ has a Borel orientation with **out-degree** at most 1, then $\mathrm{asdim}_B(X, \rho_{\mathscr{G}}) \leq 1$.

Sketch of proof. Use the proposition below, since $\mathscr{G} = \mathscr{G}_f$ for

$$f(x) = \begin{cases} y & \text{if } xR_\rightarrow y \\ x & \text{otherwise} \end{cases}$$

Proposition (Conley-Jackson-Marks-Seward-Tucker-Drob, 2023)

Let $f\colon X \to X$ be a bounded-to-one Borel function, and let $\mathscr{G} = (X, R_f)$ with $xR_f y$ if and only if $f(x) = y$ or $f(y) = x$. Then $\mathrm{asdim}_B(X, \rho_{\mathscr{G}_f}) \leq 1$

Partial Borel orientations

Miller (2008): there are Borel acyclic hyperfinite graphs where a Borel end-selection is not possible.

Proposition

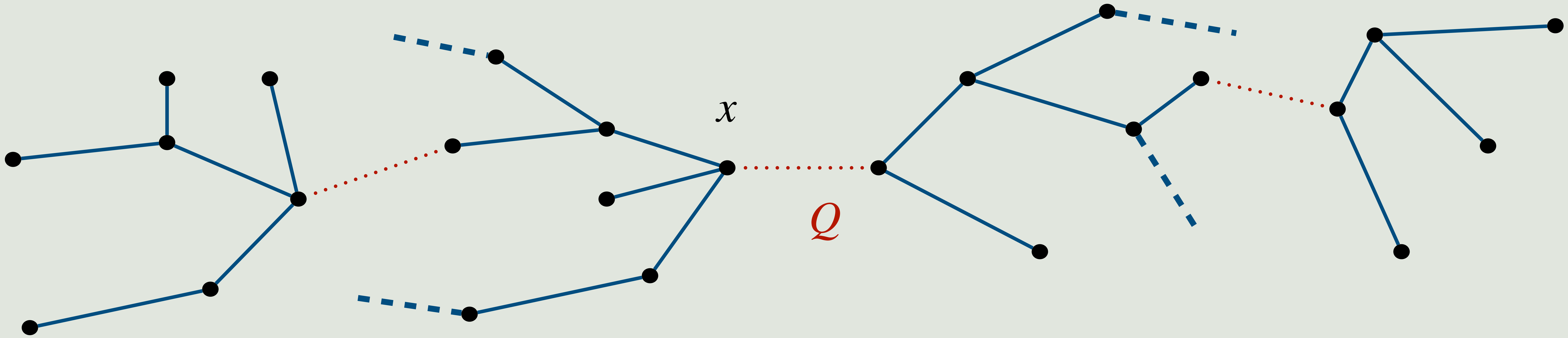
Let $\mathscr{G} = (X, R)$ be a Borel graph with bounded degree. Suppose that for every $r > 0$ there is a Borel symmetric subset $Q \subseteq R$ such that

- $\rho_{\mathscr{G}}(q_0, q_1) > r$, whenever $q_0, q_1 \in Q$ are distinct.
- $(X, R \backslash Q)$ has a Borel orientation with out-degree at most 1.

Then $\text{asdim}_B(X, \rho_{\mathscr{G}}) \leq 3$.

Sketch of proof.

Fix $r > 0$.



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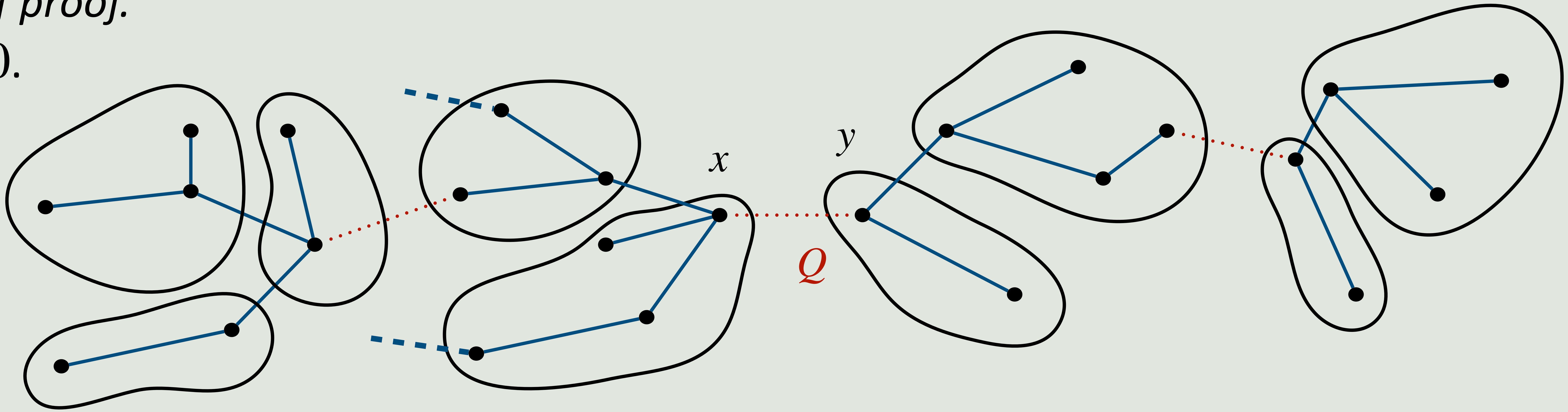

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Sketch of proof.

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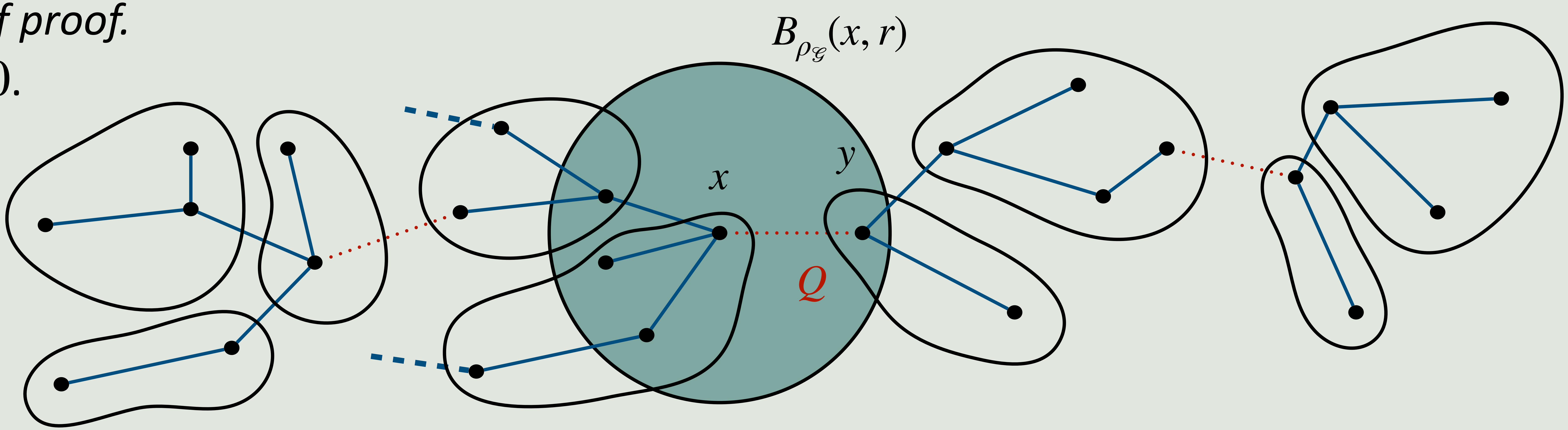


Let $\mathcal{G} = (X, R)$ be an acyclic Borel graph that has bounded degree and that is u-amenable. Fix $t > 0$ and let

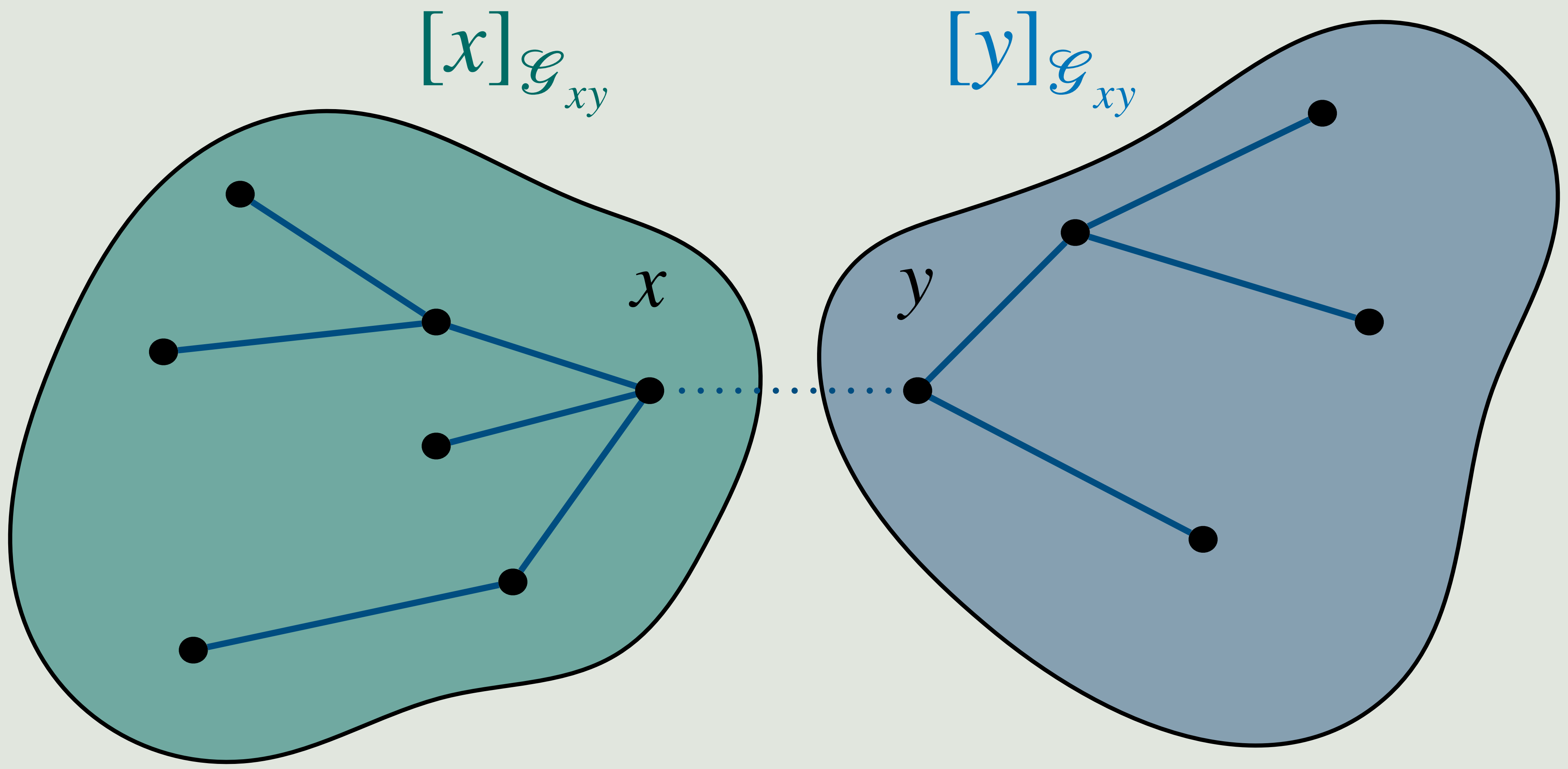
$$\lambda \colon E_{\mathcal{G}} \to \mathbb{R}_{\geq 0}$$

be a Borel function such that

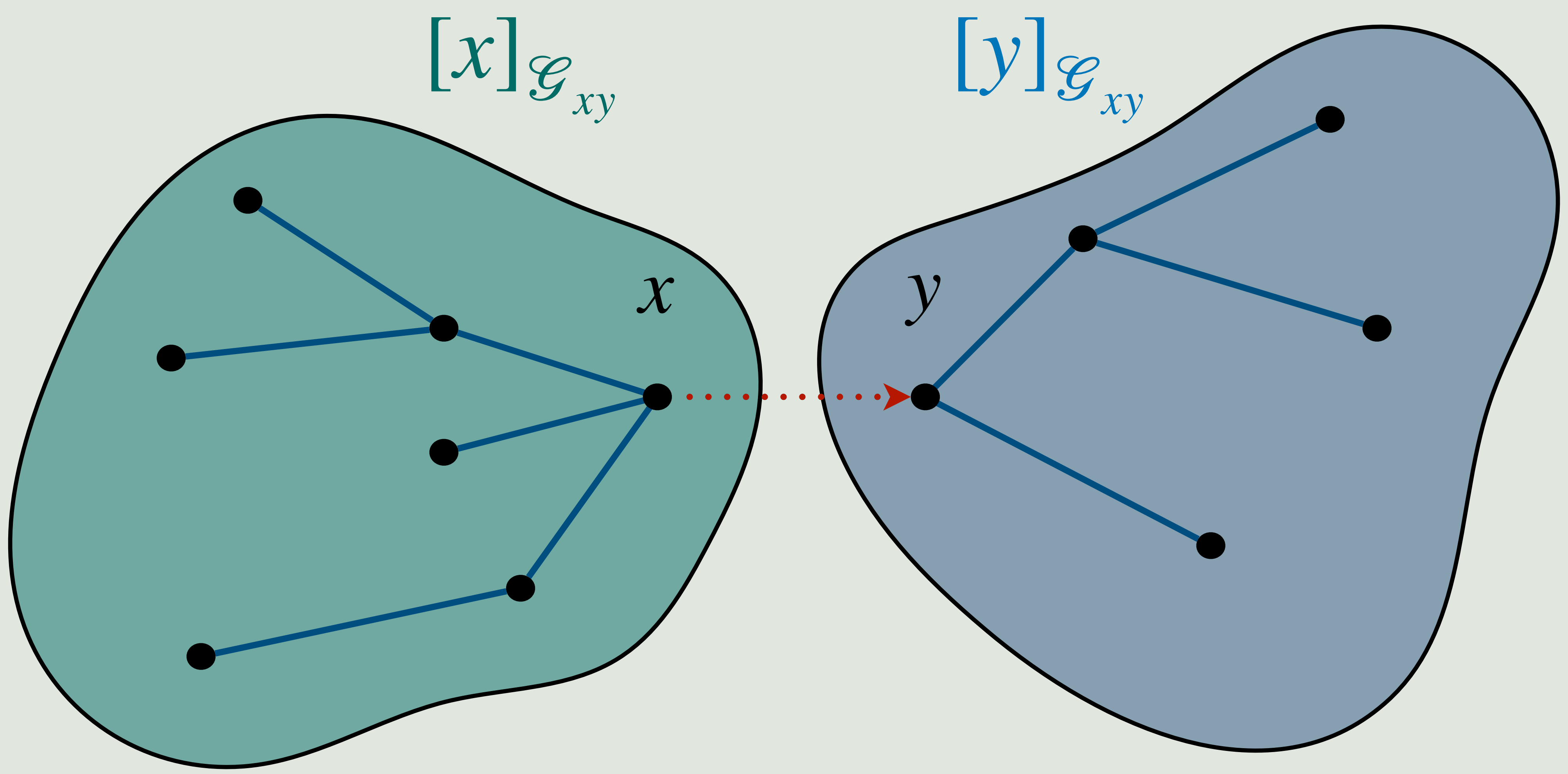
1. $\lambda_x \in \ell^1([x]_E)$ and $\|\lambda_x\|_1 = 1$,
2. $\|\lambda_x - \lambda_y\|_1 < \dfrac{1}{12}$ if $\rho_{\mathcal{G}}(x, y) < t + 2$.

Let xRy, set $\mathcal{G}_{xy} := (X, R \backslash \{(x, y), (y, x)\})$, and let $\lambda \in \{\lambda_x, \lambda_y\}$.

Then $\lambda([x]_{\mathcal{G}_{xy}}) + \lambda([y]_{\mathcal{G}_{xy}}) = 1$.



Defining a partial orientation



Set

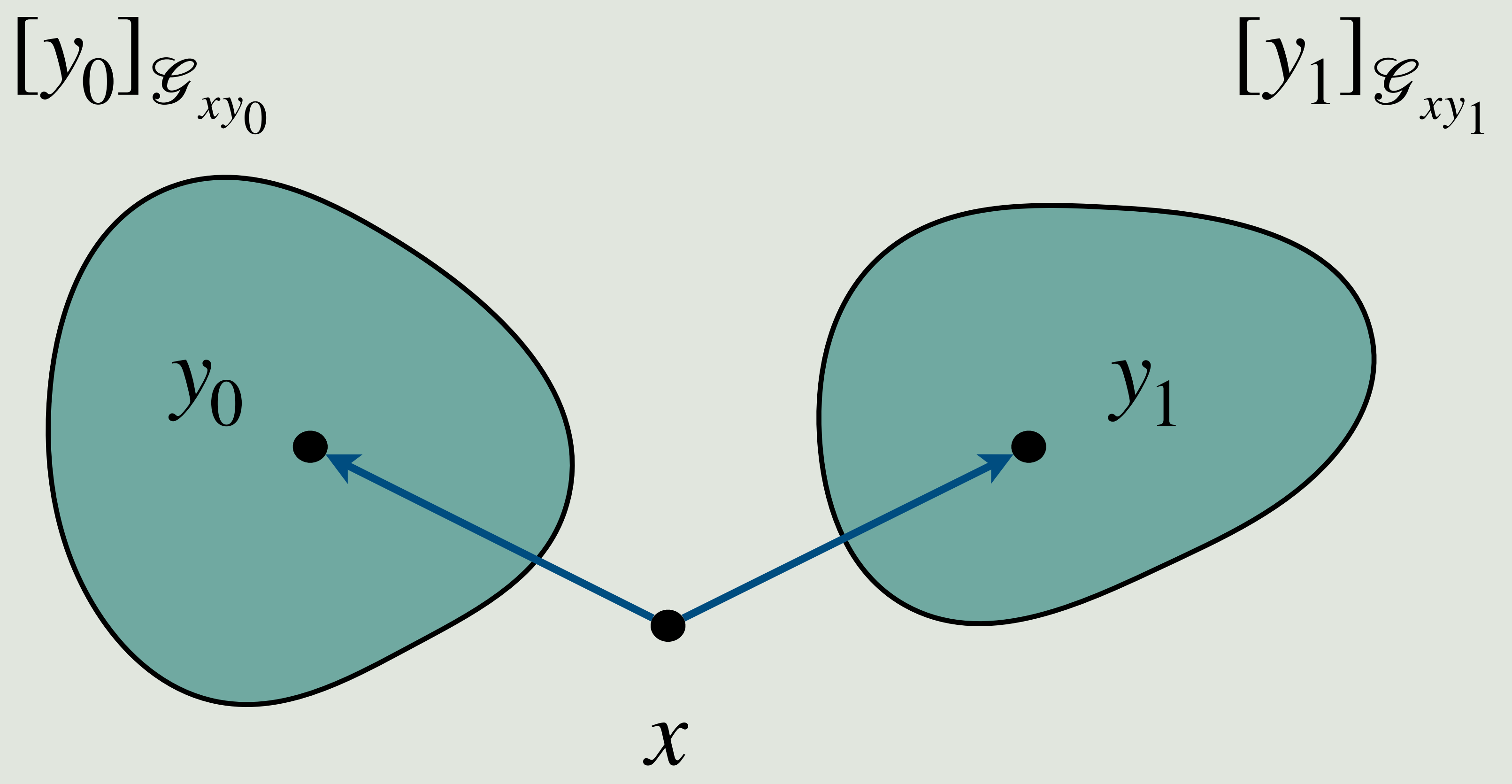
$$R_0 := \left\{ (x, y) \in R : \lambda_x([x]_{\mathcal{G}_{xy}}) \text{ and } \lambda_y([y]_{\mathcal{G}_{xy}}) \notin \left[\frac{5}{12}, \frac{7}{12} \right] \right\}.$$

Orient $x R_0^{\rightarrow} y$ if and only if $\lambda([x]_{\mathcal{G}_{xy}}) < \lambda([y]_{\mathcal{G}_{xy}})$.

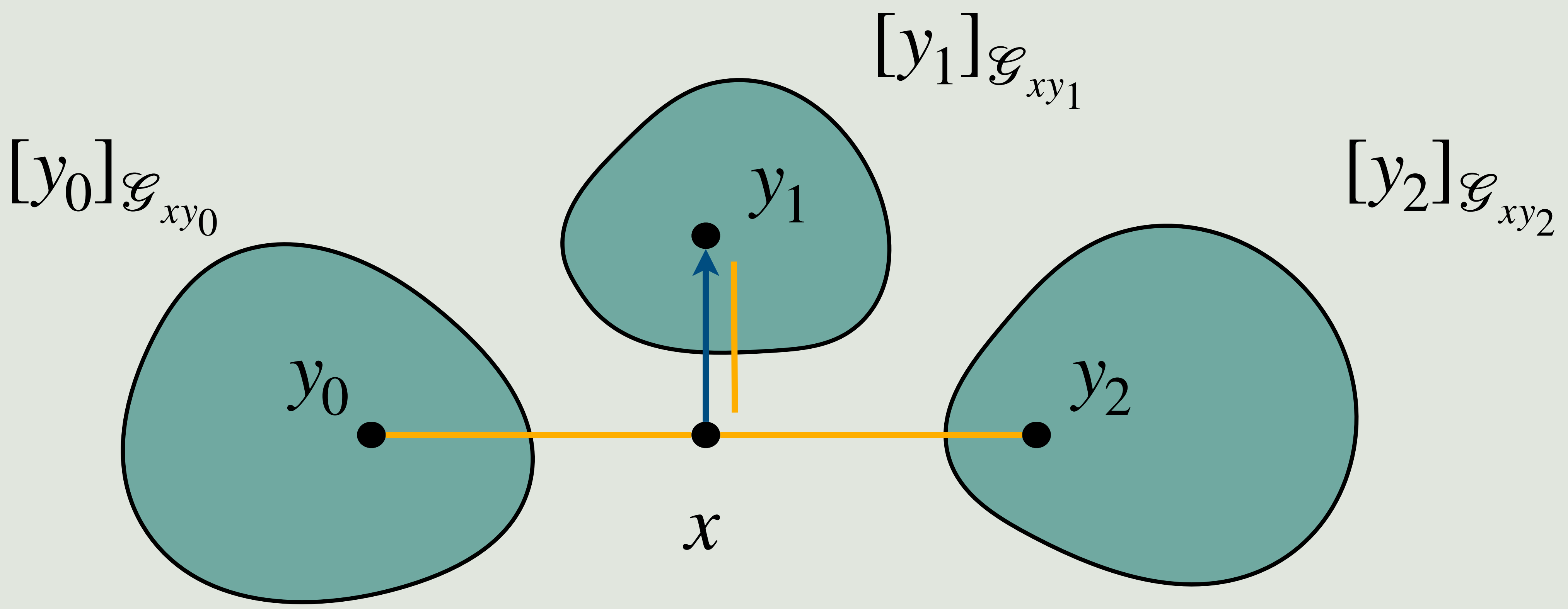
Some nice properties

The partial orientation $R_0^{\rightarrow}$ has two nice properties:

- $R_0^{\rightarrow}$ has out-degree at most 1, otherwise
  $\lambda_x([y_0]_{\mathcal{G}_{xy_0}}) + \lambda_x([y_1]_{\mathcal{G}_{xy_1}}) > 1.$



- Set $R_1 := R \backslash R_0$. Then
  $\deg_{R_1}(x) + \text{out-deg}_{R_0^{\rightarrow}}(x) \leq 2,$ otherwise
  $\lambda_x([y_0]_{\mathcal{G}_{xy_0}}) + \lambda_x([y_1]_{\mathcal{G}_{xy_1}}) + \lambda_x([y_2]_{\mathcal{G}_{xy_2}}) > 1$

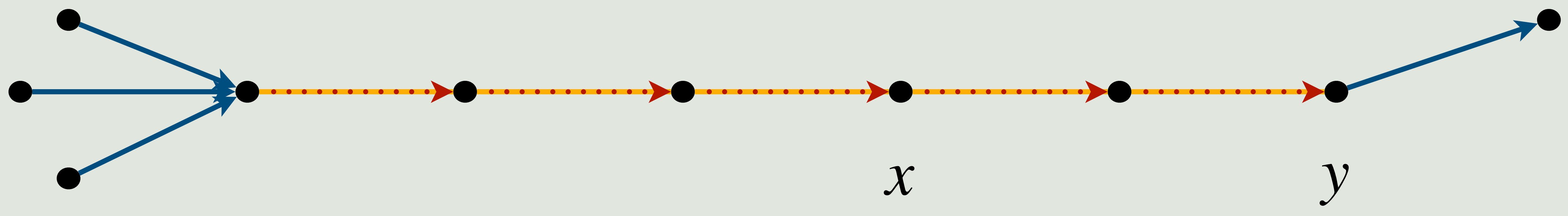


We split the edges R into two parts $R_0 \sqcup R_1$:

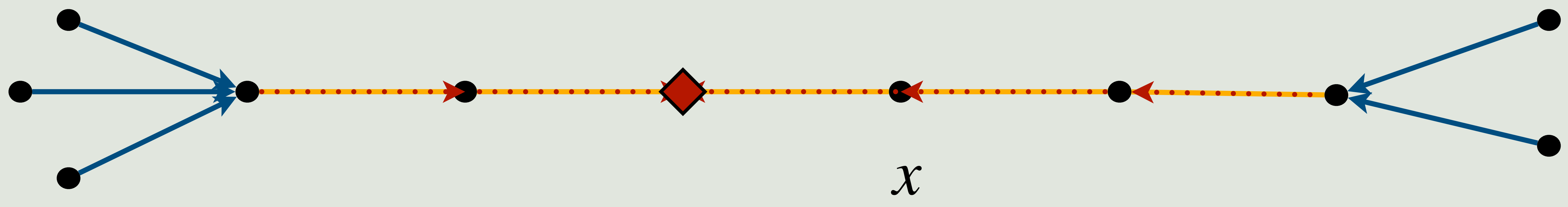
- $\mathcal{G}_0 := (X, R_0)$ has Borel orientation with out-degree at most 1, hence $\text{asdim}_B(X, \rho_{\mathcal{G}_0}) \leq 1$.
- $\mathcal{G}_1 := (X, R_1)$ has degree at most 2, and up to removing a Borel (and sufficiently sparse) set of edges, call it R_1'', we can assume that all the connected components of $\mathcal{G}_1' := (X, R_1')$, where $R_1' := R_1 \backslash R_1''$, are finite.

We aim to expand the orientation to R_1':

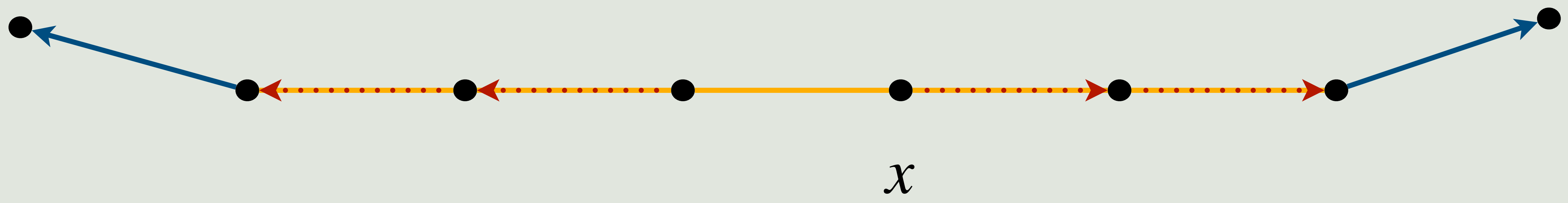
- **Case 1:** $[x]_{\mathcal{G}_1'}$ has one leaf y with out-$\text{deg}_{\vec{R_0}}(y) = 1$.



- **Case 2:** $[x]_{\mathcal{G}_1'}$ has no leaves with out-degree 1.



- **Case 3:** Both leaves in $[x]_{\mathcal{G}_1'}$ have out-degree 1.



Theorem (Naryshkin-V., 2025)

Let $\mathscr{G} = (X, R)$ be a Borel acyclic graph with bounded degree. If $\mathscr{G}$ is u-amenable, then $\text{asdim}_B(X, \rho_{\mathscr{G}}) < \infty$. In particular $E_{\mathscr{G}}$ is hyperfinite.

Summary of proof.

- Use the measure provided by amenability to define an orientation with out-degree at most 1, pointing towards vertices whose side of the graph is *heavier.*
- The left-over will be composed of lines. Expand the orientation on the non-oriented part still keeping out-degree at most 1.

On the oriented part $\mathscr{G}' = (X, R')$ we know that $\text{asdim}_B(X, \rho_{\mathscr{G}'}) \leq 1$.

- The remaining part consists of a very sparse set (depending on $t > 0$) of edges.

This suffices to deduce that $\text{asdim}_B(X, \rho_{\mathscr{G}}) \leq 3$.

Thank you!