

Spectral Theory and Descriptive Combinatorics for Borel PMP Graphs

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Caltech Logic Seminar

- The goal of classical spectral graph theory:

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- **Our goal:** Use operator theory to extract descriptive combinatorial information about infinite (bounded-degree Borel pmp) graphs.

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- **Chromatic number, $\chi(\mathcal{G})$:** The least number of colors needed for a coloring.

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$$\chi(\mathcal{G}) \leq \Delta(\mathcal{G}) + 1.$$

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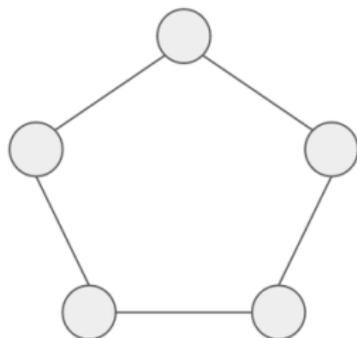
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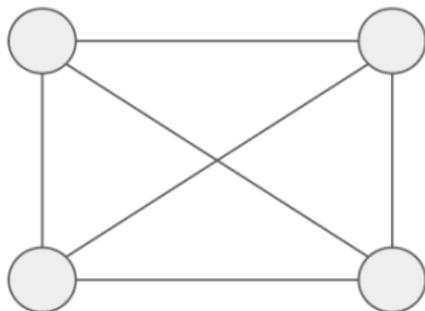
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Complete graphs

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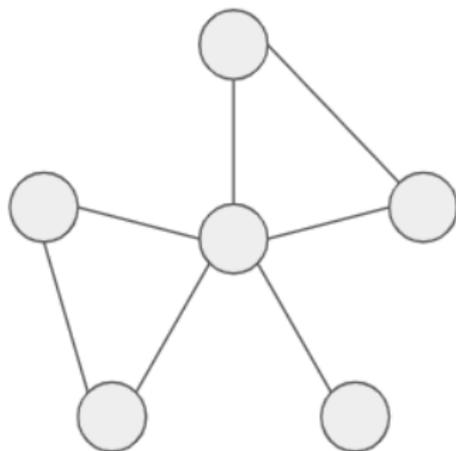
- 1 If $\Delta(\mathcal{G}) = 2$, then unless \mathcal{G} contains odd cycles, $\chi(\mathcal{G}) \leq 2$.
- 2 If $\Delta(\mathcal{G}) = d$, $d \geq 3$, then unless \mathcal{G} contains the complete graph on $d + 1$ vertices, $\chi(\mathcal{G}) \leq d$.

Brooks's Theorem

The Brooks bound is less informative for non-regular graphs:

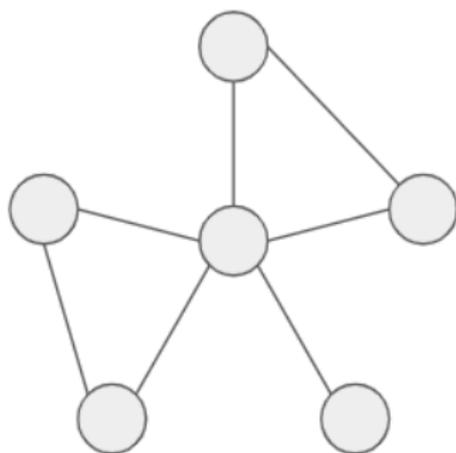
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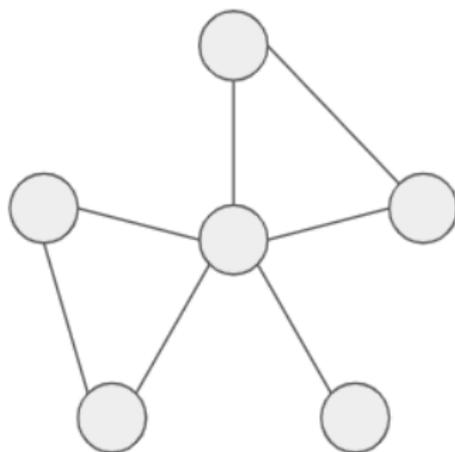
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Here $\chi(\mathcal{G}) = 3$, but Brooks's theorem gives $\chi(\mathcal{G}) \leq 5$.
The one high-degree vertex strongly influences the Brooks bound.

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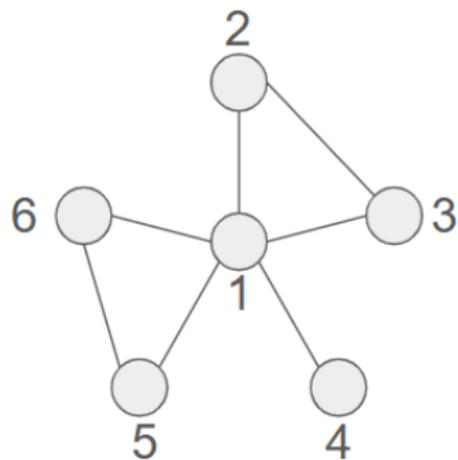
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- By the **spectral theorem**, all eigenvalues of the adjacency matrix are real.

Classical Spectral Graph Theory

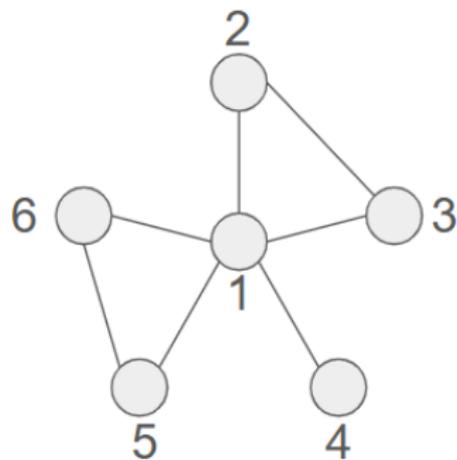


\mathcal{G}

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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$$\sigma(\mathcal{G}) = \{2.7, 1, 0.2, -1, -1, -1.9\}$$

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- If $E(\mathcal{G}) \neq \emptyset$, then $\mu_1 > 0$ and $\mu_n < 0$.
- (Perron–Frobenius) $\mu_1 \geq |\mu_i|$ for all $i \leq n$.

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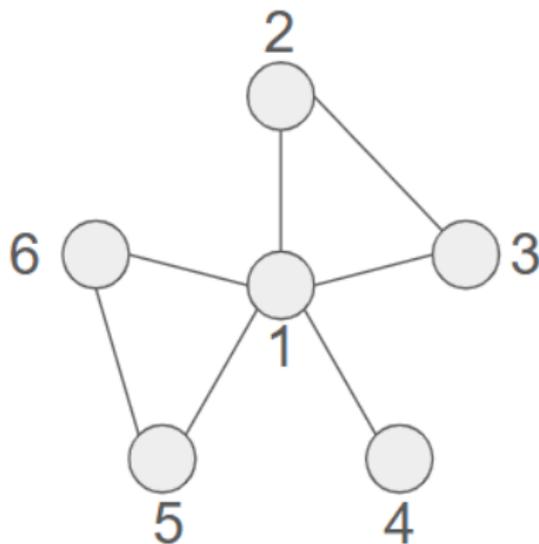
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When \mathcal{G} is non-regular: We can improve on the Brooks bound.

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$\chi(\mathcal{G}) = 3$, $\mu_1 \approx 2.7$. Brooks: $\chi(\mathcal{G}) \leq 5$, Wilf: $\chi(\mathcal{G}) \leq 3$.

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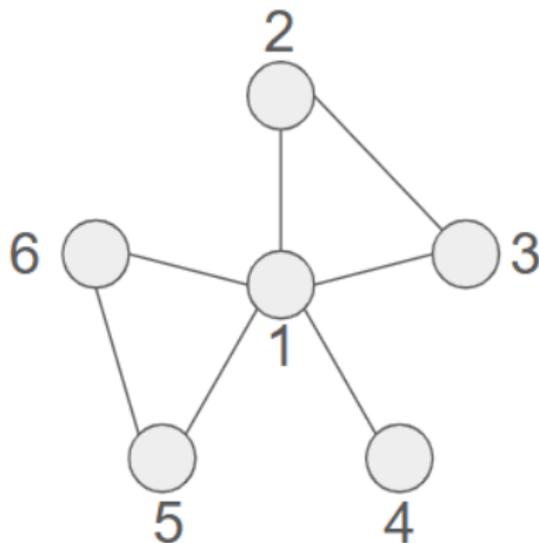
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Combining Wilf's theorem and Hoffman's theorem, we can in some cases exactly compute the chromatic number.

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- **Example:** Let Γ be a countable discrete group with finite generating set S , and let $\cdot : \Gamma \times X \rightarrow X$ be a Borel measure-preserving group action, i.e.:

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Then the Schreier graph $\mathcal{G}(\Gamma, X, S)$ is a Borel pmp graph.

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The *adjacency operator* $T_{\mathcal{G}} : L^2(X, \mu) \rightarrow L^2(X, \mu)$ of \mathcal{G} is given by

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Write $M(T_{\mathcal{G}}) = \max(\sigma(\mathcal{G}))$ and $m(T_{\mathcal{G}}) = \min(\sigma(\mathcal{G}))$.

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- 1 Put $k = \chi_{\mu}(\mathcal{G})$, and partition \mathcal{G} into measurable independent sets A_1, \dots, A_k .

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Proof sketch. **First**, to get just $\chi_{\mu}(\mathcal{G}) \geq \lceil 1 - \frac{M(T_{\mathcal{G}})}{m(T_{\mathcal{G}})} \rceil$:

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- 2 Show by induction on k that

$$(k-1)m(T_{\mathcal{G}}) + M(T_{\mathcal{G}}) \leq \sum_{1 \leq i \leq k} M(T_{ii}) = 0.$$

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- ① Ergodicity (and d -regularity): Implies that 1 is the only eigenfunction for d .
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- Can we say anything about the mcp (rather than pmp) setting?

Thank you!