The Quasi-Isometry Relation for Finitely Generated Groups

Simon Thomas

Rutgers University

21st October 2026

Why View Groups as Metric Spaces?

According to Gromov:

- Introduces concepts, constructions and ideas unimaginable in the world of pure algebra.
- Changes the class of essential examples.
- "The actual reason why one approaches a problem from a geometric angle is because one's mind is bent this way. No amount of rationalization can conceal the truth."

The Word Metric on the Cayley Graph

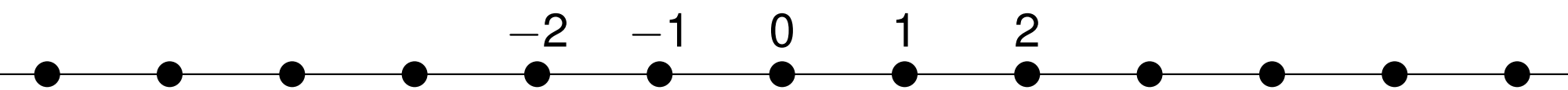
Definition

If G is a f.g. group and $S \subseteq G \setminus 1$ is a finite generating set, then the Cayley graph $\mathrm{Cay}(G, S)$ *is the graph with vertex set G and edge set*

$$E = \{\{x, y\} \mid y = xs \text{ for some } s \in S \cup S^{-1}\}.$$

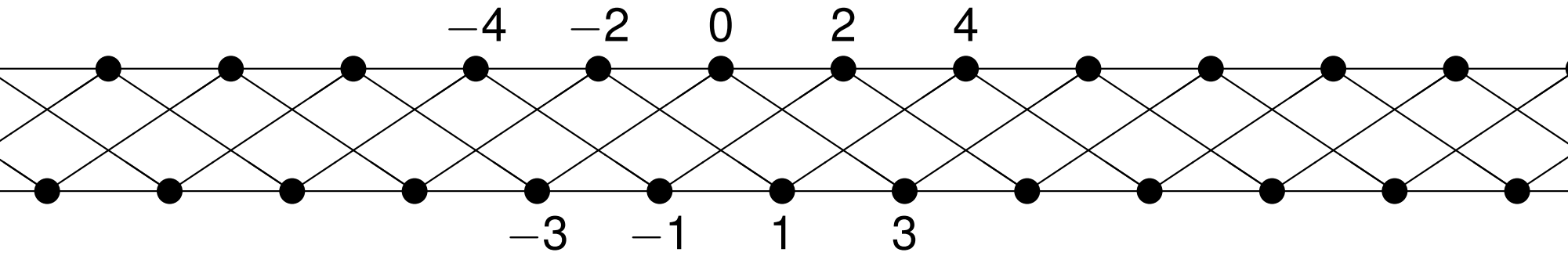
The corresponding word metric is denoted by d_S.

For example, when $G = \mathbb{Z}$ and $S = \{1\}$, then the corresponding Cayley graph is:



But which Cayley graph?

However, when $G = \mathbb{Z}$ and $S = \{2, 3\}$, then $\mathrm{Cay}(G, S)$ is:

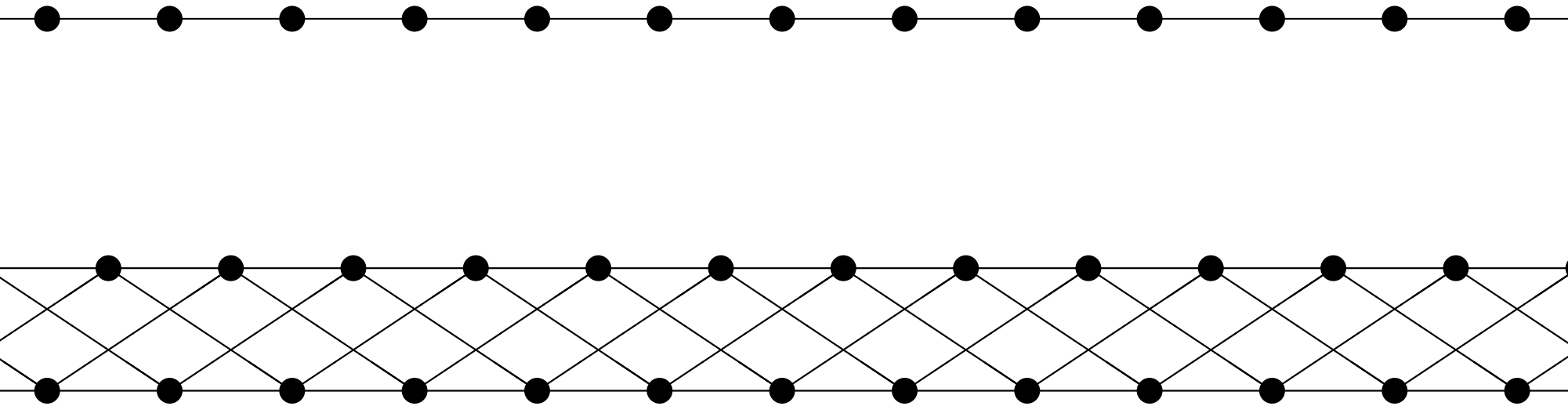


Theorem (S.T. 2004)

There does not exist an Borel choice of generators for each f.g. group which has the property that isomorphic groups are assigned isomorphic Cayley graphs.

The basic idea of geometric group theory

Although the Cayley graphs of a f.g. group G with respect to different generating sets S are usually nonisomorphic, they always have the same large scale geometry.



The quasi-isometry relation

Definition (Gromov)

Let G, H be f.g. groups with word metrics d_S, d_T respectively. Then G, H are said to be quasi-isometric, written $G \approx_{QI} H$, if there exist

- *constants $\lambda \geq 1$ and $C \geq 0$, and*
- *a map $\varphi : G \to H$*

such that for all $x, y \in G$,

$$\frac{1}{\lambda} d_S(x, y) - C \leq d_T(\varphi(x), \varphi(y)) \leq \lambda d_S(x, y) + C;$$

and for all $z \in H$,

$$d_T(z, \varphi[G]) \leq C.$$

The case when $C = 0$

Definition

Let G, H be f.g. groups with word metrics d_S, d_T respectively and let $\lambda \geq 1$. Then G, H are said to be λ-Lipschitz equivalent if there exists a bijection $\varphi : G \to H$ such that for all $x, y \in G$,

$$\frac{1}{\lambda} d_S(x, y) \leq d_T(\varphi(x), \varphi(y)) \leq \lambda d_S(x, y).$$

Theorem (Whyte 1999)

If G, H are f.g. nonamenable, then G, H are quasi-isometric if and only if G, H are Lipschitz equivalent.

The quasi-isometry relation

Definition (Gromov)

Let G, H be f.g. groups with word metrics d_S, d_T respectively. Then G, H are said to be quasi-isometric, written $G \approx_{QI} H$, if there exist

- *constants $\lambda \geq 1$ and $C \geq 0$, and*
- *a map $\varphi : G \to H$*

such that for all $x, y \in G$,

$$\frac{1}{\lambda} d_S(x, y) - C \leq d_T(\varphi(x), \varphi(y)) \leq \lambda d_S(x, y) + C;$$

and for all $z \in H$,

$$d_T(z, \varphi[G]) \leq C.$$

The quasi-isometry relation

Definition (Gromov)

Let G, H be f.g. groups with word metrics d_S, d_T respectively. Then G, H are said to be quasi-isometric, written $G \approx_{QI} H$, if there exist

- *constants $\lambda \geq 1$ and $C \geq 0$, and*
- *a map $\varphi : G \to H$*

such that for all $x, y \in G$,

$$\frac{1}{\lambda} d_S(x, y) - C \leq d_T(\varphi(x), \varphi(y)) \leq \lambda d_S(x, y) + C;$$

and for all $z \in H$,

$$d_T(z, \varphi[G]) \leq C.$$

The quasi-isometry relation

Definition (Gromov)

Let G, H be f.g. groups with word metrics d_S, d_T respectively. Then G, H are said to be quasi-isometric, written $G \approx_{QI} H$, if there exist

- *constants $\lambda \geq 1$ and $C \geq 0$, and*
- *a map $\varphi : G \to H$*

such that for all $x, y \in G$,

$$\frac{1}{\lambda} d_S(x, y) - C \leq d_T(\varphi(x), \varphi(y)) \leq \lambda d_S(x, y) + C;$$

and for all $z \in H$,

$$d_T(z, \varphi[G]) \leq C.$$

As expected ...

Observation

If S, S' are finite generating sets for the f.g. group G, then

$$id : \langle G, d_S \rangle \to \langle G, d_{S'} \rangle$$

is a λ-Lipschitz equivalence for

$$\lambda = \max\left(\{ d_{S'}(1, s) \mid s \in S \} \cup \{ d_S(1, s') \mid s' \in S' \} \right).$$

Thus while it doesn't make sense to talk about the isomorphism type of "the Cayley graph of G", it does make sense to talk about the quasi-isometry type.

The complexity of the quasi-isometry relation

Notation

Throughout $\mathcal{G}$ denotes the Polish space of (marked) f.g. groups.

Conjecture

The quasi-isometry relation $\approx_{QI}$ on $\mathcal{G}$ is strictly more complex than the isomorphism relation $\cong_{\mathcal{G}}$ with respect to Borel reducibility.

More precisely ...

The Main Conjecture

The quasi-isometry relation $\approx_{QI}$ on $\mathcal{G}$ is a universal $\mathbf{K}_\sigma$ equivalence relation.

The Polish space of (marked) f.g. groups

- A marked group $(G, \bar{s})$ consists of a f.g. group with a distinguished sequence $\bar{s} = (s_1, \cdots, s_m)$ of generators.
- For each $m \geq 1$, let $\mathcal{G}_m$ be the set of isomorphism types of marked groups $(G, (s_1, \cdots, s_m))$ with m distinguished generators.
- Let $\mathbb{F}_m$ be the free group on $\{ x_1, \cdots, x_m \}$.
- If $(G, (s_1, \cdots, s_m))$ is a marked group, we can define a homomorphism $\theta_{(G,\bar{s})} : \mathbb{F}_m \to G$ by $x_i \mapsto s_i$.
- The marked groups $(G, \bar{s})$, $(G', \bar{s}')$ are isomorphic if and only if $\ker \theta_{(G,\bar{s})} = \ker \theta_{(G',\bar{s}')}$.
- Thus $\mathcal{G}_m$ is essentially the compact space of normal subgroups of the free group $\mathbb{F}_m$.

The Polish space of (marked) f.g. groups

- Each marked group $(G, (s_1, \cdots, s_m))$ comes equipped with the corresponding word metric d_S, where $S = \{ s_1, \cdots, s_m \} \setminus 1$.
- Let $(G, \bar{s}) \in \mathcal{G}_m$ and let d_S be the corresponding word metric. For each $\ell \geq 1$, let

$$B_\ell(G, \bar{s}) = \{g \in G \mid d_S(g, 1_G) \leq \ell\}.$$

- The basic open neighborhoods of $(G, \bar{s})$ in $\mathcal{G}_m$ are given by

$$U_{(G,\bar{s}),\ell} = \{ (H, \bar{t}) \in \mathcal{G}_m \mid B_\ell(H, \bar{t}) \cong B_\ell(G, \bar{s}) \}, \qquad \ell \geq 1.$$

- There exists a canonical embedding $\mathcal{G}_m \hookrightarrow \mathcal{G}_{m+1}$ defined by

$$(G, (s_1, \cdots, s_m)) \mapsto (G, (s_1, \cdots, s_m, 1_G)).$$

- And $\mathcal{G} = \bigcup \mathcal{G}_m$ is the σ-compact Polish space of f.g. groups.

$\mathbf{K}_\sigma$ equivalence relations

Definition

The equivalence relation E on the Polish space X is $\mathbf{K}_\sigma$ if E is the union of countably many compact subsets of $X \times X$.

Example

The following are $\mathbf{K}_\sigma$ equivalence relations on the space $\mathcal{G}$ of finitely generated groups:

- the isomorphism relation $\cong$
- the quasi-isometry relation $\approx_{QI}$

Why is the quasi-isometry relation $\mathbf{K}_\sigma$?

- It is enough to show that each $\approx_{QI} \restriction \mathcal{G}_m$ is $\mathbf{K}_\sigma$.
- For each pair of integers $\lambda \geq 1$, $C \geq 0$, let $R_{\lambda,C}$ consist of the pairs
$$((G, \bar{s}), (H, \bar{t})) \in \mathcal{G}_m \times \mathcal{G}_m$$
such that there exists a (λ, C)-quasi-isometry $\varphi : G \to H$.
- If $((G, \bar{s}), (H, \bar{t})) \notin R_{\lambda,C}$, then there exists an obstruction in some balls $B_\ell(G, \bar{s})$, $B_{\lambda\ell+C}(H, \bar{t})$ for some $\ell \geq 1$.
- Thus $R_{\lambda,C}$ is a closed subset of the compact $\mathcal{G}_m \times \mathcal{G}_m$.

Some universal $\mathbf{K}_\sigma$ equivalence relations

Theorem (Kechris 1994)

There exists a universal $\mathbf{K}_\sigma$ equivalence relation $E_{\mathbf{K}_\sigma}$.

Theorem (Rosendal 2003)

The Lipschitz homeomorphism relation on the space of compact separable metric spaces is Borel bireducible with $E_{\mathbf{K}_\sigma}$.

Theorem (S.T. 2008)

The quasi-isometry relation on the space of connected 4-regular graphs is Borel bireducible with $E_{\mathbf{K}_\sigma}$.

The complexity of the quasi-isometry relation

Theorem (S.T. 2004)

The quasi-isometry relation $\approx_{QI}$ on the space $\mathcal{G}$ of f.g. groups is not smooth.

Making essential use of techniques developed by Harvey Friedman and Robin Tucker-Drob:

Main Theorem (S.T. 2026)

The quasi-isometry relation $\approx_{QI}$ does not Borel reduce to an essentially free countable Borel equivalence relation.

Essentially free relations

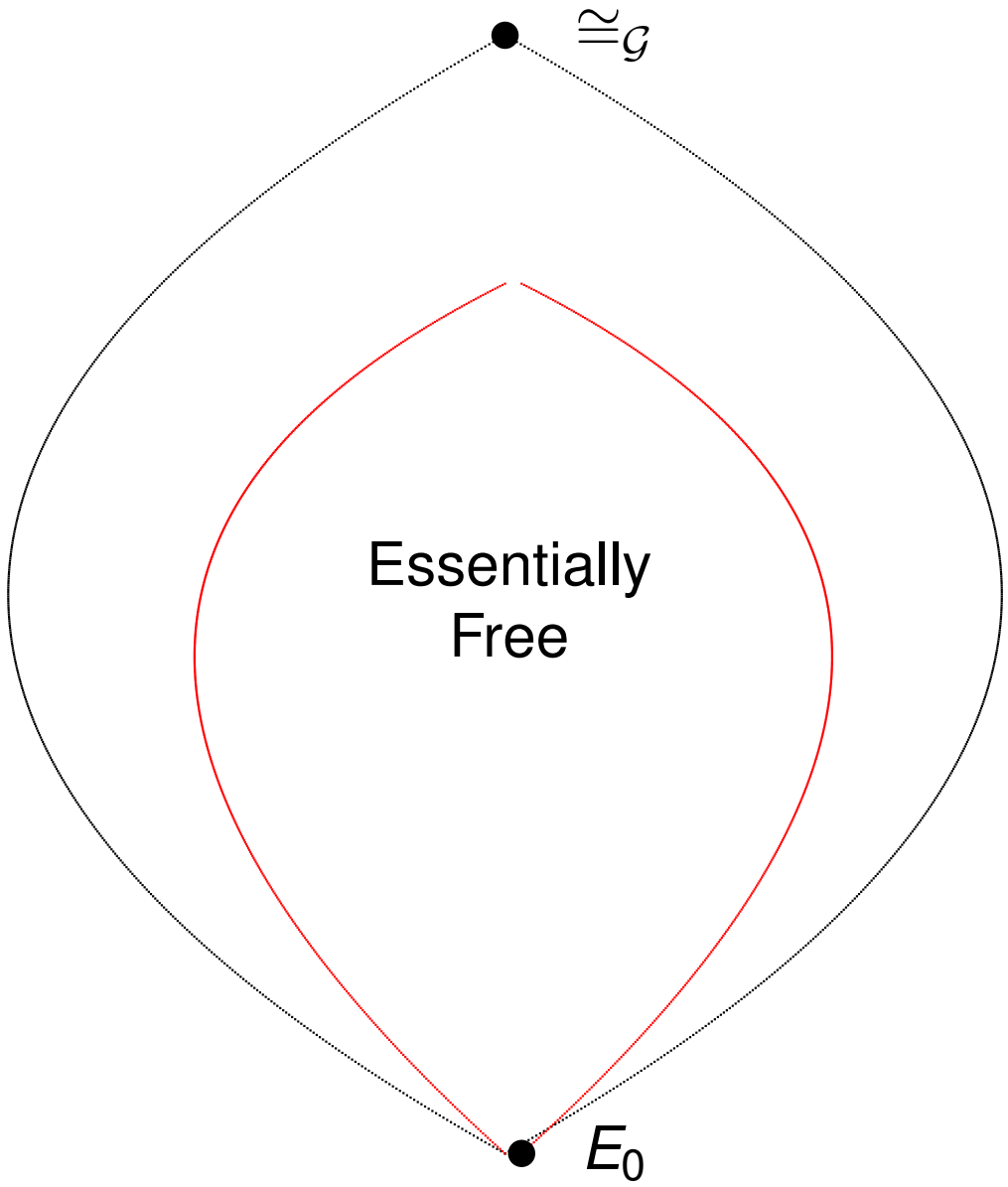
Definition

- *The countable Borel equivalence relation E on X is free if there exists a countable group G with a free Borel action on X such that $E_G^X = E$.*
- *The countable Borel equivalence relation E is essentially free if there exists a free countable Borel equivalence relation F such that $E \leq_B F$.*

Theorem (S.T. 2006)

$\cong_{\mathcal{G}}$ is not essentially free.

Countable Borel Equivalence Relations



Turing Reducibility

Convention

Throughout this talk, the powerset $\mathcal{P}(\mathbb{N})$ will be identified with $2^{\mathbb{N}}$ by identifying subsets of $\mathbb{N}$ with their characteristic functions.

Definition

If $A, B \in 2^{\mathbb{N}}$, then A is Turing reducible to B, written $A \leq_T B$, if there exists a B-oracle Turing machine which computes A.

Definition

If $A, B \in 2^{\mathbb{N}}$, then A is Turing equivalent to B, written $A \equiv_T B$, if both $A \leq_T B$ and $B \leq_T A$.

The Turing Degree of a Finitely Generated Group

For each $k \geq 1$, fix an effective enumeration $\{ w_n(x_1, \cdots, x_k) \mid n \in \mathbb{N} \}$ of the words in $x_1, \cdots, x_k$.

Definition

If $G \in \mathcal{G}$ and $s_1, \cdots, s_k$ is a finite generating set, then

$$R_G = \{ n \in \mathbb{N} \mid w_n(s_1, \cdots, s_k) = 1 \}.$$

Observation

The Turing degree of R_G does not depend on the choice $s_1, \cdots, s_k$ of generating set.

The Friedman Embedding Theorem

Theorem (Friedman 2007)

There exists a Borel map $A \mapsto F_A$ from $2^{\mathbb{N}}$ to $\mathcal{G}_2$ with the following properties:

(i) *If $A \equiv_T B$, then $F_A \cong F_B$.*

(ii) *If $H \in \mathcal{G}$ satisfies $R_H \leq_T A$, then $H \hookrightarrow F_A$.*

Remark

- The proof of the theorem shows that $A'' \leq_T R_{F_A}$.
- The map $A \mapsto F_A$ is almost certainly a Borel reduction from $\equiv_T$ to $\cong_{\mathcal{G}}$.

A Stronger Conjecture

Conjecture

The map $A \mapsto F_A$ is a Borel reduction from $\equiv_T$ to $\approx_{QI}$.

Since $\equiv_T$ is weakly universal and essentially free countable Borel equivalence relations are not weakly universal, this implies:

Theorem (S.T. 2026)

The quasi-isometry relation $\approx_{QI}$ does not Borel reduce to an essentially free countable Borel equivalence relation.

Remark

Since $\mathbb{F}_2 \hookrightarrow F_A$, it follows that $F_A \approx_{QI} F_B$ if and only if F_A, F_B are Lipschitz equivalent.

Lipschitz Embeddings

Definition

Let $G, H \in \mathcal{G}$ with word metrics d_S, d_T and let $j \geq 1$. Then an injective map $\varphi : G \to H$ is a j-Lipschitz embedding if for all $x, y \in G$,

$$d_T(\varphi(x), \varphi(y)) \leq j d_S(x, y).$$

Example

Suppose that $G, H \in \mathcal{G}$ have word metrics d_S, d_T and that $\varphi : G \to H$ is a group embedding. Then φ is j-Lipschitz for

$$j = \max\{ d_T(1, \varphi(s)) \mid s \in S \}.$$

Lipschitz Embeddings

Making essential use of techniques developed by Robin Tucker-Drob:

Theorem (S.T. 2026)

Let $G \in \mathcal{G}$ and let $\mathcal{C}$ be the set of $B \in 2^{\mathbb{N}}$ with the property that there exists $H \in \mathcal{G}$ with $R_H \leq_T B$ such that H does not Lipschitz embed into G. Then:

- $\mu(\mathcal{C}) = 1$;
- $\mathcal{C}$ *is comeager;*
- $\mathcal{C}$ *contains the cone* $\{ B \in 2^{\mathbb{N}} \mid R_G \leq_T B \}$.

Remark

If $B \in \mathcal{C}$, then F_B does not Lipschitz embed into G.

Martin's Measure

Definition

If E is a countable Borel equivalence relation on the standard Borel space X, then $\equiv_T$ is E-m-ergodic if for every Borel homomorphism $\varphi : 2^{\mathbb{N}} \to X$ from $\equiv_T$ to E, there exists a cone $\mathcal{C} \subseteq 2^{\mathbb{N}}$ such that φ maps $\mathcal{C}$ into a single E-class.

Example (Martin)

Using Borel Determinacy, it follows that $\equiv_T$ is Id_X-m-ergodic.

Remark

Martin's Conjecture implies that if E is not weakly universal, then $\equiv_T$ is E-m-ergodic.

Martin's Measure

Theorem (S.T. 2026)

If $\equiv_T$ is E-m-ergodic, then $\approx_{QI}$ does not Borel reduce to E.

- Suppose that $\psi : \mathcal{G} \to X$ is a Borel reduction from $\approx_{QI}$ to E.
- Let $\theta : 2^{\mathbb{N}} \to \mathcal{G}$ be the Borel homomorphism from $\equiv_T$ to $\cong_{\mathcal{G}}$ defined by $A \mapsto F_A$.
- Letting $\varphi = \psi \circ \theta : 2^{\mathbb{N}} \to X$, there exists a cone $\mathcal{D}$ such that φ sends $\mathcal{D}$ to a single E-class.
- It follows that θ sends $\mathcal{D}$ to a single Lipschitz equivalence class.
- But if $A \in \mathcal{D}$, then

$$\left\{ B \in 2^{\mathbb{N}} \mid F_B \text{ does not Lipschitz embed into } F_A \right\}.$$

contains a cone, and so we obtain a contradiction.

Popa Superrigidity

Notation

If G is a countably infinite group, then E_G denotes the orbit equivalence relation of the measure-preserving action $G \curvearrowright (2^G, \mu)$.

Definition

If F is a countable Borel equivalence relation on the standard Borel space X, then E_G is F-ergodic if for every Borel homomorphism $\varphi : 2^G \to X$ from E_G to F, there exists a Borel subset $Z \subseteq 2^G$ such that φ maps Z into a single F-class.

Popa Superrigidity

The following is an easy consequence of the Popa Superrigidity Theorem.

Theorem (S.T. 2009)

If $H \curvearrowright X$ is a free Borel action of a countable group H on a Polish space X, then there exists a f.g. group G such that E_G is E_H^X-ergodic.

Corollary (S.T. 2009)

If $H \curvearrowright X$ is a free Borel action of a countable group H on a Polish space X, then E_H^X is not weakly universal.

Proof of Main Theorem

Main Theorem (S.T. 2026)

The quasi-isometry relation $\approx_{QI}$ does not Borel reduce to an essentially free countable Borel equivalence relation.

- Suppose that $H \curvearrowright X$ is a free Borel action of a countable group H on a Polish space X and that $\psi : \mathcal{G} \to X$ is a Borel reduction from $\approx_{QI}$ to E_H^X.
- Let $G \in \mathcal{G}$ be such that E_G is E_H^X-ergodic.
- Then we can suppose that $G = \langle \mathbb{N}, *, 0 \rangle$.
- Let $R \in 2^{\mathbb{N}}$ be such that every left multiplication map $m \mapsto n * m$ is R-computable.

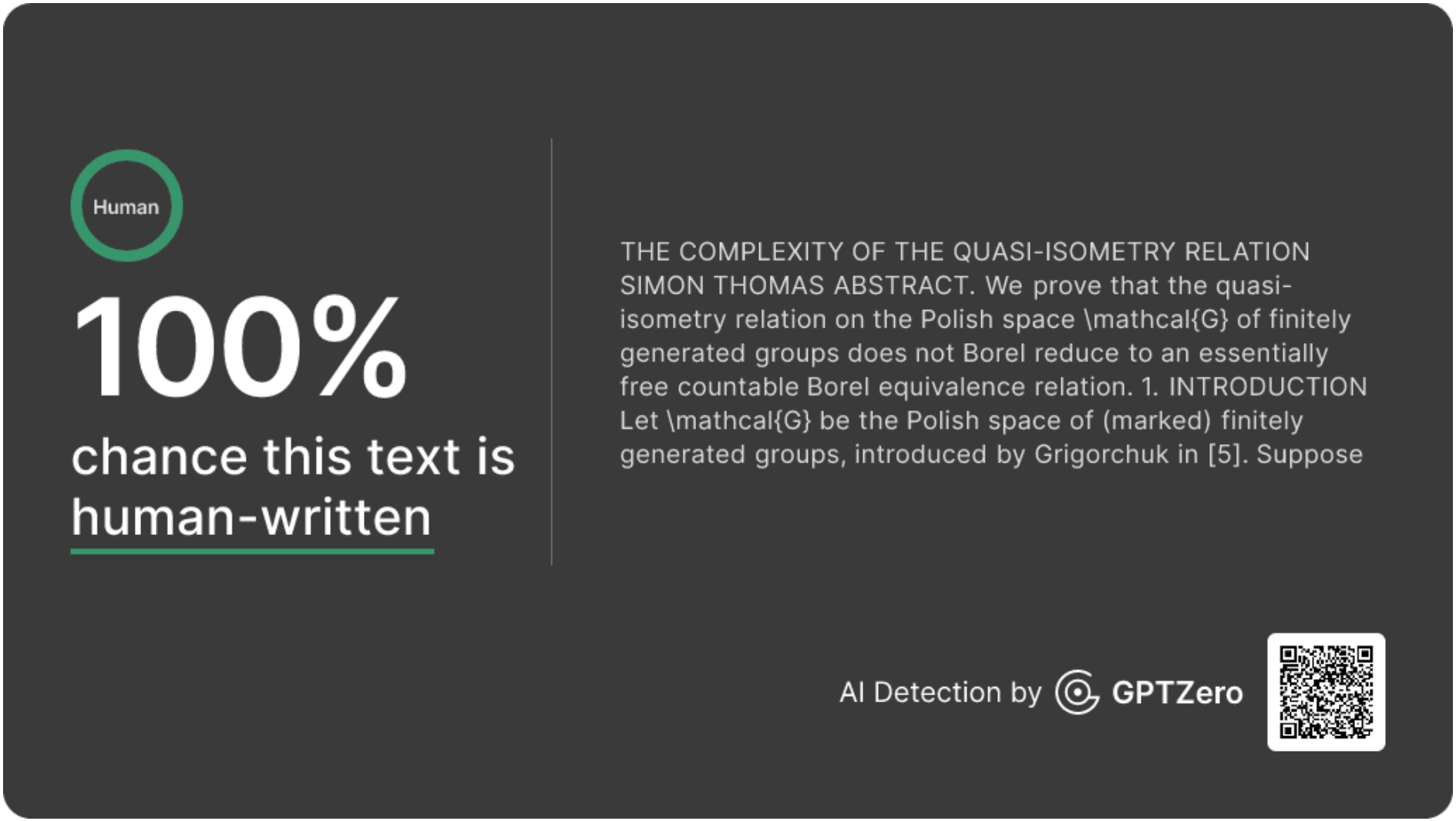
Proof of Main Theorem

- Let $\theta : 2^G = 2^{\mathbb{N}} \to \mathcal{G}$ be the Borel map defined by $A \mapsto F_{A \oplus R}$.
- If $A \in 2^{\mathbb{N}}$ and $g \in G$, then $A \oplus R \equiv_T g(A) \oplus R$.
- Thus θ is a Borel homomorphism from E_G to $\cong_{\mathcal{G}}$.
- And so $\varphi = \psi \circ \theta$ is a Borel homomorphism from E_G to E_H^X.
- Hence there exists a Borel subset $Z \subseteq 2^{\mathbb{N}}$ with $\mu(Z) = 1$ such that the groups $\{ F_{A \oplus R} \mid A \in Z \}$ lie in a single Lipschitz equivalence class.

Proof of Main Theorem

- Let $A \in Z$ and let $\mathcal{C}$ be the set of $B \in 2^{\mathbb{N}}$ with the property that there exists $H \in \mathcal{G}$ with $R_H \leq_T B$ such that H does not Lipschitz embed into $F_{A \oplus R}$.
- Then $\mu(\mathcal{C}) = 1$.
- Let $B \in Z \cap \mathcal{C}$ and let $H \in \mathcal{G}$ with $R_H \leq_T B$ such that H does not Lipschitz embed into $F_{A \oplus R}$.
- Since $R_H \leq_T B \leq_T B \oplus R$, it follows that $H \hookrightarrow F_{B \oplus R}$.
- And so $F_{B \oplus R}$ does not Lipschitz embed into $F_{A \oplus R}$, which is a contradiction.

Details can be found in:



Breaking News ...

In one of 7 papers, written since August 2026, totalling over 300 pages:

Theorem (Lupini October 2026)

$\cong_{\mathcal{G}} <_B \approx_{VI}$

Use of artificial intelligence:

ChatGPT by OpenAI, OpenAI Codex, and Claude by Anthropic were used in October 2026 during the preparation and revision of this manuscript for exploration, source retrieval, editorial and structural revision, and to assist in checking proofs, notation, terminology, internal consistency, cross-references, and bibliographic citations.

Breaking News ...



Let's try a little artificial intelligence ...

- Me: Prove that the quasi-isometry relation for finitely generated groups is complete $\mathbf{K}_\sigma$.
- Claude: The result is known (due to Ding and Gao), and the lower bound is a substantial research-level construction that I can't reproduce in full here.
- Me: Which Ding-Gao paper?
- Claude: I can't give you a reliable citation, and I shouldn't have named the paper so confidently. I believe Longyun Ding and Su Gao work on the Borel complexity of equivalence relations in this area, but I can't recall the exact title, journal, or year of a paper proving that QI of finitely generated groups is complete $\mathbf{K}_\sigma$. I may also be misattributing the result.
- Me: Sigh ...