Hyperfiniteness of (bdd-to-one) actions of commutative monoids

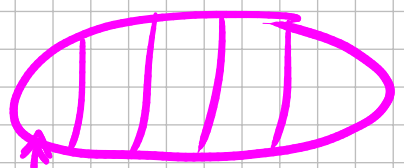
(w/ Felix Weilacher + Jing Yu).

we need this ↑

Borel equivalence relations

X std Borel space $(\mathbb{R}, S^1, 2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}})$

E eq rel on X

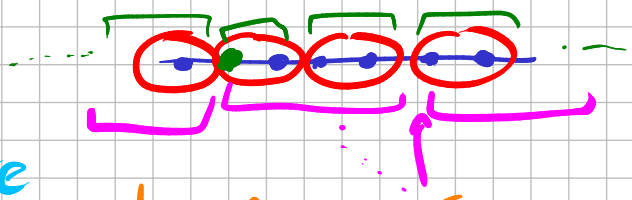


$\Gamma \curvearrowright X$

$\mathbb{Z} \curvearrowright S^1$

Hyperfinite:

$E = \bigcup_n F_n$ ← every class is finite



- Γ Borel $\curvearrowright X$ that give h/f eq. rel $\bigcup_n S_n = S_{fin}$
  - Γ locally finite (modus ponens)
    - $\mathbb{Z}$ (Slaman-Steel)



$\bigcup \mathbb{Z}^n$
- $\mathbb{Z}^n$ (Weiss)

$\mathbb{Z}^{\oplus \omega}$ = $\bigcup \mathbb{Z}^n$
- f.g. of poly growth (Jackson-Kechris-Louveau)
  - abelian (Gao-Jackson)
- locally of poly growth (Conley-Jackson-Marks-Seward-Tucker-Drob)
- polycyclic → (includes some exp growth)

$\mathbb{Z}/2 \wr \mathbb{Z}$.

· ~~Conjecturally~~ open question(ly), all amenable gps ???
(Weiss)

solvable ???

(reason: Γ amenable $\Rightarrow$ $\Gamma \curvearrowright X$ eq rel is "amenable")

CJMSTD showed:

Borel asymptotic dim $\leq d < \infty$ $\Longrightarrow$ hyperfinite

Short

of a group action

$\forall s \in \mathbb{N}$ you can solve some local coloring problem (at scale s)

OTOH: M amenable monoid

$M \curvearrowright X$ has amenable eq rel.

hyperfinite ??? (Dougherty-Jackson-Kechris)

Known: $\mathbb{N} \curvearrowright X$ always hypersmooth (basically hyperfinite).

· Q: M commutative monoid

$M \curvearrowright X$ h/f?

open question!

already open for $\mathbb{N}^2$????

Thm (小波 - Weilacher - 于)

If M is a ctble commutative monoid,

$M \curvearrowright X$ is a bounded-to-one Borel action,

then its orbit eq rel is h/f.

Thm (us): M f.g. comm monoid.

$M \curvearrowright X$ bdd-to-one ???

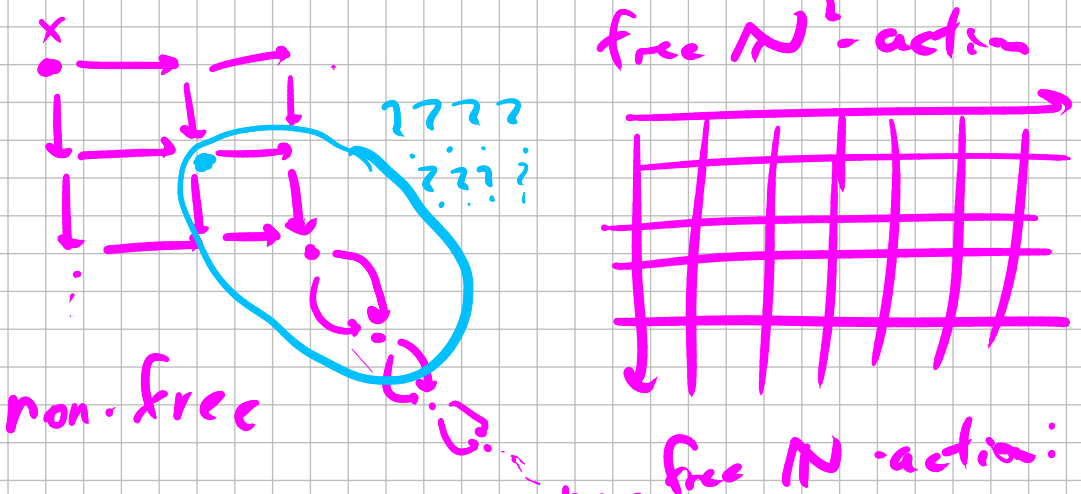
$\Rightarrow$ free

Then $\mathrm{asdim}_B(M \curvearrowright X)$ is finite.

$x \to mx$ has bdd fibers



CJMST.D: increasing union of finite asdim] great!
"Borel asdim" is h/f.

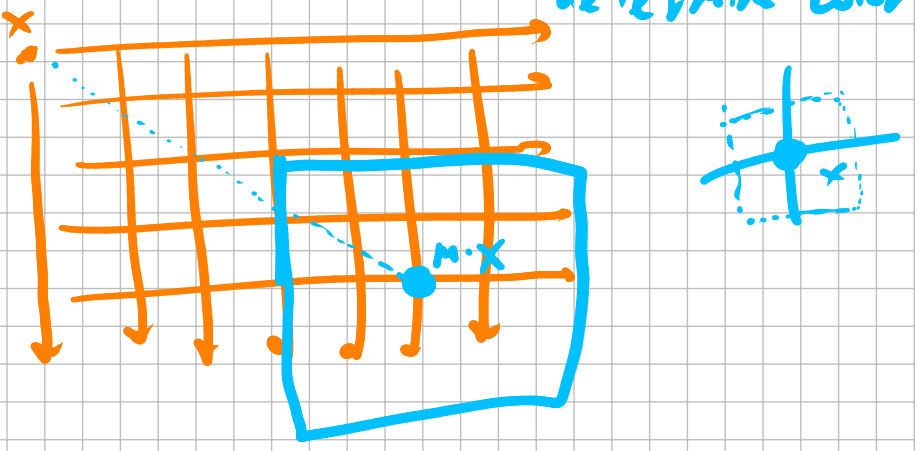


Pf for free actions: of $\mathbb{N}^2$! ✓

We already know CJMST.D that $\mathbb{Z}^2$ has finite ~~Borel~~ asdim

cts LOCAL(oblgn) ← only you need to look at a finite window to determine color

Then in $\mathbb{N}^2$:
x looks at a point in front, runs the local alg there.



- for non-f.g.: don't need finite asdim for inc. union ↑

"Borel dimension group ...." → slow dimension ($\log(s)$ dim) ↑ weak dim. poly(s) dim)

- an LLM proved that it's always finite d.m (even for non-free) (Wang)
- try doing this for a cts action on a cpt space

$M \curvearrowright X$



done by finite union bound.

Lesson: use compactness!!